Study On the Hyperbola $9x^2 - 7y^2 = 8$

1M.A.Gopalan, 2J. Shanthi, 3S.Vidhyalakshmi
1Professor, 2Assistant Professor, 3Assistant Professor
Shrimati Indira Gandhi College

***Abstract* - The hyperbola represented by the binary quadratic equation 9x^2-7y^2=8 is analyzed for finding its non-zero distinct integer solutions. A few interesting relations among its solutions are presented. Also, knowing an integral solution of the given hyperbola, integer solutions for other choices of hyperbolas and parabolas are presented. The formulation of second order Ramanujan Numbers with base numbers as real integers and Gaussian integers is illustrated.**

***keywords* - Hyperbola, pell-like equation, non-homogeneous quadratic, integer solutions, second order Ramanujan number**

I. INTRODUCTION

The binary quadratic Diophantine equations of the form $ax^2 - by^2 = N, (a,b,N \neq 0)$ are rich in variety and have been analyzed by many mathematicians for their respective integer solutions for particular values of a, b and N. In this context, one may refer [1-14].

This communication concerns with the problem of obtaining non-zero distinct integer solutions to the binary quadratic equation given by $9x^2 - 7y^2 = 8$ representing hyperbola. A few interesting relations among its solutions are presented. Knowing an integral solution of the given hyperbola, integer solutions for other choices of hyperbolas and parabolas are presented. The formulation of second order Ramanujan Numbers with base numbers as real integers and Gaussian integers is illustrated.

II. METHOD OF ANALYSIS

The Diophantine equation representing the binary quadratic to be solved for its non-zero distinct integer solutions is

$$9x^2 - 7y^2 = 8 \tag{1}$$

Consider the linear transformations

$$x = X + 7T\ ,\ y = X + 9T \tag{2}$$

From (1) and (2), we have

$$X^2 = 63T^2 + 4 \tag{3}$$

whose smallest positive integer solution is

$$X_0 = 16\ ,\ T_0 = 2$$

To obtain the other solutions of (3) , consider the pell equation

$$X^2 = 63T^2 + 1 \tag{4}$$

whose smallest positive integer solution is $\left(\widetilde{X}_0\ ,\ \widetilde{T}_0\right) = (16\ ,\ 2)$

The general solution of (4) is given by

$$\widetilde{T}_n = \frac{1}{2\sqrt{63}} g_n\ ,\ \widetilde{X}_n = \frac{1}{2} f_n$$

where

$$f_n = \left(8+\sqrt{63}\right)^{n+1} + \left(8-\sqrt{63}\right)^{n+1}$$

$$g_n = \left(8+\sqrt{63}\right)^{n+1} - \left(8-\sqrt{63}\right)^{n+1}\ ,\ n = 0,1,2,3.....$$

Applying Brahmagupta lemma between $(X_0\ ,T_0)$ and $\left(\widetilde{X}_n\ ,\widetilde{T}_n\right)$ we have

$$T_{n+1} = T_0\,\widetilde{X}_n + X_0\,\widetilde{T}_n$$

$$X_{n+1} = X_0\,\widetilde{X}_n + D\,T_0\,\widetilde{T}_n$$

$$\Rightarrow T_{n+1} = f_n + \frac{8}{\sqrt{63}} g_n \tag{5}$$

$$X_{n+1} = 8f_n + \frac{63}{\sqrt{63}} g_n \tag{6}$$

Substituting (5) and (6) in (2) , the other integer solutions of (3) are given by

$$x_{n+1} = 15 f_n + \frac{119}{\sqrt{63}} g_n \tag{7}$$

$$y_{n+1} = 17 f_n + \frac{135}{\sqrt{63}} g_n \tag{8}$$

Replacing n by $n+1$ in (7), we get

$$x_{n+2} = 15 f_{n+1} + \frac{119}{\sqrt{63}} g_{n+1}$$

$$= 15\left(8 f_n + \sqrt{63} g_n\right) + \frac{119}{\sqrt{63}}\left(8 g_n + \sqrt{63} f_n\right) \tag{9}$$

$$x_{n+2} = 239 f_n + \frac{1897}{\sqrt{63}} g_n$$

Replacing n by $n+1$ in (9) , we get

$$x_{n+3} = 239 f_{n+1} + \frac{1897}{\sqrt{63}} g_{n+1}$$

$$= 239\left(8 f_n + \sqrt{63} g_n\right) + \frac{1897}{\sqrt{63}}\left(8 g_n + \sqrt{63} f_n\right) \tag{10}$$

$$x_{n+3} = 3809 f_n + \frac{30233}{\sqrt{63}} g_n$$

Eliminating f_n , g_n between (7), (9) and (10) , we have

$$x_{n+1} - 16 x_{n+2} + x_{n+3} = 0 \ , \ n = -1, 0, 1, 2, ... \tag{11}$$

In a similar manner, one obtains

$$y_{n+2} = 271 f_n + \frac{2151}{\sqrt{63}} g_n \tag{12}$$

$$y_{n+3} = 4319 f_n + \frac{34281}{\sqrt{63}} g_n \tag{13}$$

Eliminating f_n , g_n between (8) , (12) and (13) , we have

$$y_{n+1} - 16 y_{n+2} + y_{n+3} = 0 \ , \ n = -1, 0, 1, 2, 3 \ ,...... \tag{14}$$

Thus, (11) and (14) represent the recurrence relations satisfied by the value of x and y respectively,

Some numerical examples of x_n and y_n satisfying (1) are given in the Table 1 below:

Table: 1 Numerical Example

n	x_{n+1}	y_{n+1}
-1	30	34
0	478	542
1	7618	8638
2	121410	137666
3	1934942	2194018

From the above table, we observe some interesting relations among the solutions which are presented below:

Both x_n and y_n values are even.

One can generate second order Ramanujan numbers by choosing x and y values suitably.

For illustration, consider

$$\begin{aligned} x_3 &= 121410 \\ &= 121410*1 = 60705*2 = 40470*3 = 24282*5 = 20235*6 \\ &= 13490*9 = 12141*10 = 8094*15 = 6745*18 = 6390*19 \\ &= 4047*30 \end{aligned} \tag{*}$$

Now,

$$121410*1 = 60705*2$$

$$\Rightarrow 121411^2 + 60703^2 = 121409^2 + 60707^2$$

$$= 184254815130$$

$60705*2 = 40470*3$

$\Rightarrow 60707^2 + 40467^2 = 60703^2 + 40473^2$

$= 5322917938$

$40470*3 = 24282*5$

$\Rightarrow 40473^2 + 24277^2 = 40467^2 + 24287^2$

$= 2227436458$

$24282*5 = 20235*6$

$\Rightarrow 24287^2 + 20229^2 = 24277^2 + 20241^2$

$= 999070810$

$20235*6 = 13490*9$

$\Rightarrow 20241^2 + 13481^2 = 20229^2 + 13499^2$

$= 591435442$

$13490*9 = 12141*10$

$\Rightarrow 13499^2 + 12131^2 = 13481^2 + 12151^2$

$= 329384162$

$12141*10 = 8094*15$

$\Rightarrow 12151^2 + 8079^2 = 12131^2 + 8109^2$

$= 212917042$

$8094*15 = 6745*1$

$\Rightarrow 8109^2 + 6727^2 = 8079^2 + 6763^2$

$= 111008410$

$6745*18 = 6390*19$

$\Rightarrow 6763^2 + 6371^2 = 6727^2 + 6409^2$

$= 86327810$

Thus, 18425485130, 5322917938, 2227436458, 999070810, 591435442, 329384162, 212917042, 111008410, 86327810, 57211570 are second order Ramanujan numbers whose base numbers are Real Integers. Also, from (*),

$121410*1 = 60705*2$

$\Rightarrow (121410+i)^2 + (60705-2i)^2 = (12140-i)^2 + (60705+2i)^2$

$= 18425485120$

$60705*2 = 40470*3$

$\Rightarrow (60705+2i)^2 + (40470-3i)^2 = (60705-2i)^2 + (40470+3i)^2$

$= 5322917912$

$40470*3 = 24282*5$

$\Rightarrow (40470+3i)^2 + (24282-5i)^2 = (40470-3i)^2 + (24282+5i)^2$

$= 22227436390$

$24282*5 = 20235*6$

$\Rightarrow (24282+3i)^2 + (20235-6i)^2 = (24282-3i)^2 + (20235+6i)^2$

$= 999070704$

$20235*6 = 13490*9$

$\Rightarrow (20235+6i)^2 + (13490-9i)^2 = (20235-6i)^2 + (13490+9i)^2$

$= 591435208$

$$13490*9=12141*10$$
$$\Rightarrow (13490+9i)^2+(12141-10i)^2=(12490-9i)^2+(12141+10i)^2$$
$$=329383800$$
$$12141*10=8094*15$$
$$\Rightarrow (12141+10i)^2+(8094-15i)^2=(12141-10i)^2+(8094+15i)^2$$
$$=212916392$$
$$8094*15=6745*18$$
$$\Rightarrow (8094+15i)^2+(6754-18i)^2=(8094-15i)^2+(6745+18i)^2$$
$$=111007312$$
$$6754*18=6390*19$$
$$\Rightarrow (6745+18i)^2+(6390-19i)^2=(6745-18i)^2+(6390+19i)^2$$
$$=86326440$$
$$6390*19=4047*30$$
$$\Rightarrow (6390+19i)^2+(4047-30i)^2=(6390-19i)^2+(4047+30i)^2$$
$$=57209048$$

Thus ,

18425485120 , 5322917912 , 2227436390 , 999070704 , 591435208, 329383800 , 212916392 , 111007312 , 86326440 , 57209048 are second order Ramanujan number whose base number are Gaussian Integers.

1. **Relations among the solutions are given below.**
   - $7y_{n+1}-x_{n+2}+8x_{n+1}=0$
   - $7y_{n+2}-8x_{n+2}+x_{n+1}=0$
   - $7y_{n+3}-127x_{n+2}+8x_{n+1}=0$
   - $x_{n+3}-127x_{n+1}+112y_{n+1}=0$
   - $y_{n+2}-9x_{n+1}-8y_{n+1}=0$
   - $y_{n+3}-144x_{n+1}-127y_{n+1}=0$
   - $x_{n+3}-x_{n+1}-14y_{n+2}=0$
   - $112y_{n+3}-127x_{n+3}+x_{n+1}=0$
   - $8y_{n+3}-9x_{n+1}-127y_{n+2}=0$
   - $x_{n+2}-8x_{n+3}+7y_{n+3}=0$
   - $127x_{n+2}-8x_{n+3}+7y_{n+1}=0$
   - $9x_{n+2}-8y_{n+2}+y_{n+1}=0$
   - $y_{n+3}-18y_{n+2}-y_{n+1}=0$
   - $8x_{n+2}-x_{n+3}+7y_{n+2}=0$
   - $8y_{n+1}-127y_{n+2}+9x_{n+3}=0$
   - $9x_{n+3}-8y_{n+3}+y_{n+2}=0$
   - $144x_{n+3}-127y_{n+3}+y_{n+1}=0$
   - $y_{n+1}-16y_{n+2}+y_{n+3}=0$
   - $9x_{n+2}-y_{n+3}+8y_{n+2}=0$

2. **Each of the following expressions is a nasty number:**
   - $\frac{3}{112}[30233x_{2n+2}-119x_{2n+4}+448]$
   - $3[135x_{2n+2}-119y_{2n+2}+4]$
   - $\frac{3}{8}[2151x_{2n+2}-119y_{2n+3}+32]$
   - $\frac{3}{127}[34281x_{2n+2}-119y_{2n+3}+508]$

- $\frac{3}{7}\left[30233x_{2n+3}-1897x_{2n+4}+28\right]$
- $\frac{3}{8}\left[135x_{2n+3}-1897y_{2n+2}+32\right]$
- $3\left[2151x_{2n+3}-1897y_{2n+3}+4\right]$
- $\frac{3}{8}\left[34281x_{2n+3}-1897y_{2n+4}+32\right]$
- $\frac{3}{127}\left[135x_{2n+4}-30233y_{2n+2}+508\right]$
- $\frac{3}{8}\left[2151x_{2n+4}-30233y_{2n+3}+32\right]$
- $3\left[34281x_{2n+4}-30233y_{2n+3}+32\right]$
- $3\left[135y_{2n+3}-2151y_{2n+2}+36\right]$
- $\frac{3}{144}\left[1351y_{2n+4}-34281y_{2n+2}+576\right]$
- $3\left[2151y_{2n+4}-34281y_{2n+3}+36\right]$

3. Each of the following expressions is a cubical integer:

- $\frac{1}{224}\left[30233x_{3n+3}-119x_{3n+5}+90699x_{n+1}-357x_{n+3}\right]$
- $\frac{1}{2}\left[135x_{3n+3}-119y_{3n+3}+405x_{n+1}-357y_{n+1}\right]$
- $\frac{1}{16}\left[2151x_{3n+3}-119y_{3n+5}+6453x_{n+1}-357y_{n+2}\right]$
- $\frac{1}{254}\left[34281x_{3n+3}-119y_{3n+5}+102843x_{n+1}-357y_{n+3}\right]$
- $\frac{1}{14}\left[30233x_{3n+4}-1897x_{3n+5}+90699x_{n+2}-5691x_{n+3}\right]$
- $\frac{1}{16}\left[135x_{3n+4}-1897y_{3n+3}+405x_{n+2}-5691y_{n+1}\right]$
- $\frac{1}{2}\left[2151x_{3n+4}-1897y_{3n+4}+6453x_{n+2}-5691y_{n+2}\right]$
- $\frac{1}{16}\left[34281x_{3n+4}-1897y_{3n+5}+102843x_{n+2}-5691y_{n+3}\right]$
- $\frac{1}{254}\left[135x_{3n+5}-30233y_{3n+3}+405x_{n+3}-90699y_{n+1}\right]$
- $\frac{1}{16}\left[2151x_{3n+5}-30233y_{3n+4}+6453x_{n+3}-90699y_{n+2}\right]$
- $\frac{1}{2}\left[34281x_{3n+5}-30233y_{3n+5}+102843x_{n+3}-90699y_{n+3}\right]$
- $\frac{1}{18}\left[135y_{3n+4}-2151y_{3n+3}+405y_{n+2}-6453y_{n+1}\right]$
- $\frac{1}{288}\left[135y_{3n+5}-34281y_{3n+3}+405y_{n+3}-102843y_{n+1}\right]$
- $\frac{1}{18}\left[2151y_{3n+5}-34281y_{3n+4}+6453y_{n+3}-102843y_{n+2}\right]$

4. Each of the following expressions is a bi-quadratic integer:

- $\frac{1}{224}\left[30233x_{4n+4}-119x_{4n+6}+120932x_{2n+2}-476x_{2n+4}+1344\right]$
- $\frac{1}{2}\left[135x_{4n+4}-119y_{4n+4}+540x_{2n+2}-476y_{2n+2}+12\right]$
- $\frac{1}{16}\left[2151x_{4n+4}-119y_{4n+5}+8604x_{2n+2}-476y_{2n+3}+96\right]$
- $\frac{1}{254}\left[34281x_{4n+4}-119y_{4n+6}+137124x_{2n+2}-476y_{2n+4}+1524\right]$
- $\frac{1}{14}\left[30233x_{4n+5}-1897x_{4n+6}+120932x_{2n+3}-7588x_{2n+4}+84\right]$

- $\frac{1}{16}\left[135x_{4n+5}-1897y_{4n+4}+540x_{2n+3}-7588y_{2n+2}+96\right]$
- $\frac{1}{2}\left[2151x_{4n+4}-1897y_{4n+4}+8604x_{2n+2}-7588y_{2n+2}+12\right]$
- $\frac{1}{16}\left[34281x_{4n+5}-1897y_{4n+6}+137124x_{3n+4}-7588y_{3n+5}+96\right]$
- $\frac{1}{254}\left[135x_{4n+6}-30233y_{4n+4}+540x_{2n+4}-120932y_{2n+2}+1524\right]$
- $\frac{1}{16}\left[2151x_{4n+6}-30233y_{4n+5}+8604x_{2n+4}-120932y_{2n+3}+96\right]$
- $\frac{1}{2}\left[34281x_{2n+4}-30233y_{4n+6}+137124x_{2n+4}-120932y_{2n+4}+12\right]$
- $\frac{1}{18}\left[135y_{4n+6}-2151y_{4n+4}+540y_{2n+3}-8604y_{2n+2}+108\right]$
- $\frac{1}{288}\left[135y_{4n+6}-34281y_{4n+4}+540y_{2n+4}-137124y_{2n+2}+1728\right]$
- $\frac{1}{18}\left[2151y_{4n+6}-34281y_{4n+5}+8604y_{2n+4}-137124y_{2n+3}+108\right]$

5. **Each of the following expressions is a quintic integer:**

- $\frac{1}{224}\left[\begin{array}{r}30233x_{5n+5}-119x_{5n+7}+151165x_{3n+3}-595x_{3n+5}\\+302330x_{n+1}-1190x_{n+3}\end{array}\right]$
- $\frac{1}{2}\left[\begin{array}{r}135x_{5n+5}-119y_{5n+5}+675x_{3n+3}-595y_{3n+3}\\+1350x_{n+1}-1190y_{n+1}\end{array}\right]$
- $\frac{1}{16}\left[\begin{array}{r}2151x_{5n+5}-119y_{5n+6}+10755x_{3n+5}-595y_{3n+5}\\+21510x_{n+1}-1190y_{n+2}\end{array}\right]$
- $\frac{1}{254}\left[\begin{array}{r}34281x_{5n+5}-119y_{5n+7}+171405x_{3n+3}-595y_{3n+5}\\+342810x_{n+1}-1190y_{n+3}\end{array}\right]$
- $\frac{1}{14}\left[\begin{array}{r}30233x_{5n+6}-1897x_{5n+7}+151165x_{3n+4}-9485x_{3n+5}\\+302330x_{n+2}-18970x_{n+3}\end{array}\right]$
- $\frac{1}{16}\left[\begin{array}{r}135x_{5n+6}-1897y_{5n+5}+675x_{3n+4}-9485y_{3n+3}\\+2350x_{n+2}-18970y_{n+1}\end{array}\right]$
- $\frac{1}{2}\left[\begin{array}{r}2151x_{5n+6}-1897y_{5n+6}+10755x_{3n+4}-9485y_{3n+4}\\+21510x_{n+2}-18970y_{n+2}\end{array}\right]$
- $\frac{1}{16}\left[\begin{array}{r}34281x_{3n+4}-1897y_{5n+7}+171405x_{3n+4}-9485y_{3n+5}\\+342810x_{n+2}-18970y_{n+3}\end{array}\right]$
- $\frac{1}{254}\left[\begin{array}{r}135x_{5n+7}-30233y_{5n+5}+675x_{3n+5}-151165y_{3n+3}\\+1350x_{n+3}-302330y_{n+1}\end{array}\right]$
- $\frac{1}{16}\left[\begin{array}{r}2151x_{5n+7}-30233y_{5n+6}+10755x_{3n+5}-151165y_{3n+4}\\+21510x_{n+3}-302330y_{n+2}\end{array}\right]$
- $\frac{1}{2}\left[\begin{array}{r}34281x_{5n+7}-30233y_{5n+7}+171405x_{3n+5}-151165y_{3n+5}\\+342810x_{n+3}-302330y_{n+3}\end{array}\right]$
- $\frac{1}{18}\left[\begin{array}{r}135y_{5n+6}-2151y_{5n+5}+675y_{3n+4}-10755y_{3n+3}\\+1350y_{n+2}-21510y_{n+1}\end{array}\right]$
- $\frac{1}{288}\left[\begin{array}{r}135y_{5n+7}-34281y_{5n+5}+675y_{3n+5}-171405y_{3n+3}\\+1350y_{n+3}-342810y_{n+1}\end{array}\right]$
- $\frac{1}{18}\left[\begin{array}{r}2151y_{5n+7}-34281y_{5n+6}+10755y_{3n+5}-171405y_{3n+4}\\+21510y_{n+3}-342810y_{n+2}\end{array}\right]$

REMARKABLE OBSERVATIONS

I. Employing linear combinations among the solutions of (1), one may generate integer solutions of hyperbolas. For simplicity and clear understandings the other choices of hyperbola are presented in the Table 2 below:

Table: 2 Hyperbola

S.NO	Hyperbola	(X,Y)
1	$Y^2-63X^2=784$	$\begin{pmatrix}15x_{n+2}-239x_{n+1}, \\ 1897x_{n+1}-119x_{n+2}\end{pmatrix}$
2	$Y^2-63X^2=200704$	$\begin{pmatrix}15x_{n+3}-3809x_{n+1}, \\ 30233x_{n+1}-119x_{n+3}\end{pmatrix}$
3	$Y^2-63X^2=16$	$\begin{pmatrix}15y_{n+1}-17x_{n+1}, \\ 135x_{n+1}-119y_{n+1}\end{pmatrix}$
4	$Y^2-63X^2=1024$	$\begin{pmatrix}15y_{n+2}-271x_{n+1}, \\ 2151x_{n+1}-119y_{n+2}\end{pmatrix}$
5	$Y^2-63X^2=258064$	$\begin{pmatrix}15y_{n+3}-4319x_{n+1}, \\ 34281x_{n+1}-119y_{n+3}\end{pmatrix}$
6	$Y^2-63X^2=784$	$\begin{pmatrix}239x_{n+3}-3809x_{n+2}, \\ 30233x_{n+2}-1897x_{n+3}\end{pmatrix}$
7	$Y^2-63X^2=1024$	$\begin{pmatrix}239y_{n+1}-17x_{n+2}, \\ 135x_{n+2}-1897y_{n+1}\end{pmatrix}$
8	$Y^2-63X^2=24$	$\begin{pmatrix}239y_{n+2}-271x_{n+2}, \\ 2151x_{n+2}-1897y_{n+2}\end{pmatrix}$
9	$Y^2-63X^2=1024$	$\begin{pmatrix}34281x_{n+2}-1897y_{n+3}, \\ 239y_{n+3}-4319x_{n+2}\end{pmatrix}$
10	$Y^2-63X^2=258064$	$\begin{pmatrix}3809y_{n+1}-17x_{n+3}, \\ 135x_{n+3}-30233y_{n+1}\end{pmatrix}$
11	$Y^2-63X^2=1024$	$\begin{pmatrix}3809y_{n+2}-271x_{n+3}, \\ 2151x_{n+3}-30233y_{n+2}\end{pmatrix}$
12	$Y^2-63X^2=16$	$\begin{pmatrix}3809y_{n+3}-4319x_{n+3}, \\ 34281x_{n+3}-30233y_{n+3}\end{pmatrix}$
13	$Y^2-63X^2=1296$	$\begin{pmatrix}271y_{n+1}-17y_{n+2}, \\ 135y_{n+2}-2151y_{n+1}\end{pmatrix}$
14	$Y^2-63X^2=331776$	$\begin{pmatrix}4319y_{n+1}-17y_{n+3}, \\ 135y_{n+3}-34281y_{n+1}\end{pmatrix}$
15	$Y^2-63X^2=1296$	$\begin{pmatrix}4319y_{n+2}-271y_{n+3}, \\ 2151y_{n+3}-34281y_{n+2}\end{pmatrix}$

II. Employing linear combinations among the solutions of (1), one may generate integer solutions of parabolas. For simplicity and clear understandings the other choices of parabola are presented below in Table 3:

Table: 3 Parabola

S.NO	Parabola	(X,Y)
1	$14Y-63X^2=784$	$\begin{pmatrix}15x_{n+2}-239x_{n+1}, \\ 1897x_{2n+2}-119x_{2n+3}+28\end{pmatrix}$
2	$224Y-63X^2=200704$	$\begin{pmatrix}15x_{n+3}-3809x_{n+1}, \\ 30233x_{2n+2}-119x_{2n+4}+448\end{pmatrix}$
3	$2Y-63X^2=16$	$\begin{pmatrix}15y_{n+1}-17x_{n+1}, \\ 135x_{2n+2}-119y_{2n+2}+4\end{pmatrix}$
4	$16Y-63X^2=1024$	$\begin{pmatrix}15y_{n+2}-271x_{n+1}, \\ 2151x_{2n+2}-119y_{2n+3}+32\end{pmatrix}$

5	$254Y-63X^2=258064$	$\begin{pmatrix}15y_{n+3}-4319x_{n+1}, \\ 34281x_{2n+2}-119y_{2n+4}+508\end{pmatrix}$
6	$14Y-63X^2=784$	$\begin{pmatrix}239x_{n+3}-3809x_{n+2}, \\ 30233x_{2n+3}-1897x_{2n+4}+28\end{pmatrix}$
7	$16Y-63X^2=1024$	$\begin{pmatrix}239y_{n+1}-17x_{n+2}, \\ 135x_{2n+3}-1897y_{2n+2}+32\end{pmatrix}$
8	$2Y-63X^2=16$	$\begin{pmatrix}239y_{n+2}-271x_{n+2}, \\ 2151x_{2n+3}-1897y_{2n+3}+4\end{pmatrix}$
9	$16Y-63X^2=1024$	$\begin{pmatrix}239y_{n+3}-4319x_{n+2}, \\ 34281x_{2n+3}-1897y_{2n+4}+32\end{pmatrix}$
10	$254Y-63X^2=258064$	$\begin{pmatrix}3809y_{n+1}-17x_{n+3}, \\ 135x_{2n+4}-30233y_{2n+2}+508\end{pmatrix}$
11	$16Y-63X^2=1024$	$\begin{pmatrix}3809y_{n+2}-271x_{n+3}, \\ 2151x_{2n+3}-30233y_{2n+3}+32\end{pmatrix}$
12	$2Y-63X^2=16$	$\begin{pmatrix}3809y_{n+3}-4319x_{n+3}, \\ 34281x_{2n+4}-30233y_{2n+4}+4\end{pmatrix}$
13	$18Y-63X^2=1296$	$\begin{pmatrix}271y_{n+1}-17y_{n+2}, \\ 135y_{2n+3}-2151y_{2n+2}+36\end{pmatrix}$
14	$288Y-63X^2=331776$	$\begin{pmatrix}4319y_{n+1}-17y_{n+3}, \\ 135y_{2n+4}-34281y_{2n+2}+576\end{pmatrix}$
15	$18Y-63X^2=1296$	$\begin{pmatrix}4319y_{n+2}-271y_{n+3}, \\ 2151y_{2n+4}-34281y_{2n+3}+36\end{pmatrix}$

III. CONCLUSION

In equation (1), the linear transformations for x and y may also be considered as $x=X-7T$, $y=X-9T$. Following the above procedure, one obtains different set of solutions to $9x^2-7y^2=8$. As quadratic equations are rich in variety ,the readers may search for integer solutions to other choices of hyperbolas with suitable relations among the solutions.

REFERENCES

[1] R. D. Carmichael , "The Theory of Numbers and Diophantine Analysis", Dover Publications, New York ,1950.

[2] L. E. Disckson,"History of theory of Numbers, Vol. II, Chelsea Publishing co.", New York , 1952.

[3] L. J .Mordell, "Diophantine Equations", Academic Press, London,1969.

[4] M. A. Gopalan and R. Anbuselvi, "Integral solutions of $4ay^2-(a-1)x^2=3a+1$ ", Acta Ciencia Indica,Vol.XXXIV M, No.1, Pp:291-295, 2008.

[5] Gopalan et al., "Integral points on the hyperbola $(a+2)x^2-ay^2=4a(k-1)+2k^2, a,k>0$", Indian journal of science,1(2),Pp:125-126, 2012.

[6] M. A. Gopalan, S. Devibala, R. Vidhyalakshmi, "Integral Points On the Hyperbola $2X^2-3Y^2=5$ ", American Journal of Applied Mathematics amd Mathematical Sciences, Volume 1, N0.1, Pp: 1-4, 2012.

[7] M. A. Gopalan, Sangeetha G, Manju Somanath, "Integral Points on the h Integral Points on the hyperbola $(a+2)x^2+ay^2=4a(k-1)+2k^2, a,k>0$", Indian Journal of Science Volume 1,Issue 2, Pp:125-126, 2012.

[8] S. Vidhyalakshmi, etal., "Observations on the hyperbola $ax^2-(a+1)y2=3a-1$ ", Discovery,4(10), Pp:22-24, 2013.

[9] M. A. Gopalan, S. Vidhyalakshmi and A. Kavitha, "Observations the hyperbola $10y^2-3x^2=13$ ", Archimedes J.Math.,3(1),Pp:31-34, 2013.

[10] Vijayasankar. A., Gopalan. M. A., Krithika.V., "On the binary quadratic equation $ax^2-(a+1)y^2=a$ ", World Journal of Engineering Research and Technology , Vol.3, Iss.5, Pp:473-479, September 2017.

[11] Vijayasankar. A., Gopalan. M. A., Krithika.V., "On the Hyperbola $2x^2-3y^2=15$ ", Indo – Iranian Journal of Scientific Research, Vol.1, Iss.1, Pp:144-153, Decmber 2017.

[12] Anusheela N., Shreemathi Adiga, and Gopalan M.A., "Observations on pell-like equation $11x^2 - 6y^2 = -10$", WWJMRD, vol 4(7), Pp: 16-19, Jul 2018.

[13] Shreemathi Adiga, Anusheela N. and Gopalan M.A., "Observations on the hyperbola $8x^2 - 5y^2 = 27$", IJMA, 9(9), Pp.65-73, 2018.

[14] M.A.Gopalan , Sharadha Kumar, " On the Pell-Like Equation $3x^2 - 8y^2 = 40$ ", IJMR, Vol.5, Issue 1, Page No.95-106, January 2019.