Aluthge Transformation of (n,k) Quasi Class Q and (n,k) Quasi Class Q^* Operators

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Abstract - **In this paper, a new class of operators called (n,k) quasi class Q and (n,k) quasi class Q^* operators are introduced and studied some properties. (n,k) Quasi class Q and (n,k) quasi class Q^* composition and weighted composition operators on L^2 (λ) and H^2 (β) are characterized. Also we discuss (n,k) quasi class Q and (n,k) quasi class Q^* composite multiplication operator on L^2 space and Aluthge transformation of these class of operators are obtained.**

keywords - **Class Q operators, class Q^* operators, composition operators, weighted composition operators, Aluthge transformation.**

I. Introduction

Let H be an infinite dimensional separable Complex Hilbert space. Let $B(H)$ be the algebra of all bounded linear operators acting on H. Let T be an operator on H. Every operator T can be decomposed into $T = U|T|$ with a partial isometry U, where $|T|$ is the square root of (T^*T). If U is determined uniquely by the kernel condition $N(U) = N(|T|)$, then this decomposition is called the polar decomposition, which is one of the most important results in operator theory.

Recall that an operator T is said to be paranormal if $\|Tx\|^2 \leq \|T^2x\| \, \|x\|$ for every $x \in H$ [7]. An operator T is said to be n-paranormal if $\|Tx\|^{\{n+1\}} \leq \|T^{\{n+1\}}x\|\|x^n\|$ for every $x \in H$ [17] and normaloid if $r(T) = \|T\|$, where $r(T)$ denotes the spectral radius of T. An operator T is of class Q [6], if $T^{\{*2\}}T^2 - 2T^*T + I \geq 0$. Equivalently $T \in Q$ if $\|Tx\|^2 \leq \frac{1}{2}(\|T^2x\|^2 + \|x\|^2)$ for every $x \in H$. Class Q operators are introduced and studied by B. P. Duggal et al and it is well known that every class Q operator is not necessarily normaloid and every paranormal operator is a normaloid of class Q. ie $P \subseteq Q \cap N$, where P and N denotes the class of paranormal and normaloid operators respectively. Also he showed that the restriction of T to an invariant subspace is again a class Q operator.

Devika, Suresh [4], introduced a new class of operators which we call the quasi class Q operators and it is defined as, for $T \in B(H)$
$$\|T^2x\|^2 \leq \frac{1}{2}(\|T^3x\|^2 + \|Tx\|^2) \text{ for every } x \in H$$

In [8], A k-quasi class Q operator is defined as follows, An operator T is of k-quasi class Q if
$$\left\|T^{\{k+1\}}x\right\|^2 \leq \frac{1}{2}(\left\|T^{\{k+2\}}x\right\|^2 + \|T^kx\|^2) \text{ for every } x \in H$$
and k is a natural number. D. Senthil Kumar, Prasad. T in [15], has defined the new class of operators which we call M-class Q operators. An operator T is of M class Q if for a fixed real number $M \geq 1$, T satisfies $M^2T^{\{*2\}}T^2 - 2T^*T + I \geq 0$ or equivalently $\|Tx\|^2 \leq \frac{1}{2}(M^2\|T^2x\|^2 + \|x\|^2)$ for every $x \in H$ and a fixed real number $M \geq 1$.

In [18], Youngoh Yang and Cheoul Jun Kim introduced a class Q^* operators. If $T^{\{*2\}}T^2 - 2TT^* + I \geq 0$, then T is called class Q^* operators. He also proved that if T is class Q^* if and only if $\|T^*x\|^2 \leq \frac{1}{2}(\|T^2x\|^2 + \|x\|^2)$ for every $x \in H$. In [12], D. Senthil Kumar et. al. introduced quasi class Q^* operators. If $T^{\{*3\}}T^3 - 2(T^*T)^2 + T^*T \geq 0$ \$, then T is called quasi class Q^* operators. He also proved that if T is quasi class Q^* if and only if $\|T^*Tx\|^2 \leq \frac{1}{2}(\|T^3x\|^2 + \|Tx\|^2)$ for every $x \in H$.

In this paper, we study some properties of (n,k) quasi class Q and (n,k) quasi class Q^* operators and we derive conditions for composition and weighted composition operators to be (n,k) quasi class Q and (n,k) quasi class Q^*. Aluthge transformation of (n,k) quasi class Q and (n,k) quasi class Q^* operators are derived. Conditions for Composite multiplication operators to be (n,k) quasi class Q and (n,k) quasi class Q^* are also obtained. A characterization of (n,k) quasi class Q and (n,k) quasi class Q^* composition and weighted composition operators on weighted Hardy space are obtained.

II. (n,k) Quasi Class Q Operators

In this section, we define new class of operators called (n,k) quasi class Q, which is a super class of n class Q and quasi n class Q operators and studied some properties of this class of operators.

Definition 2.1.

An operator $T \in B(H)$ is said to be (n,k) quasi class Q if for every positive integer n and for every $x \in H$

$$\|T^{\{k+1\}}x\|^2 \leq \frac{1}{1+n}\left(\|T^{\{k+1+n\}}x\|^2 + n\|T^k x\|^2\right)$$

when $n = 1$ it is of k quasi class Q operators.

Theorem 2.2.
An operator T is of (n,k) quasi class Q if and only if
$T^{\{*k\}}\left(T^{\{*1+n\}}T^{\{1+n\}} - (1+n)T^{\{*\}}T + nI\right)T^k \geq 0$ for every positive integer n.

Proof
Since T is (n,k) quasi class Q operator, we have

$$\|T^{\{k+1\}}x\|^2 \leq \frac{1}{1+n}\left(\|T^{\{k+1+n\}}x\|^2 + n\|T^k x\|^2\right)$$

$$\Leftrightarrow \|T^{\{k+1+n\}}x\|^2 - (1+n)\|T^{\{k+1\}}x\|^2 + n\|T^k x\|^2) \geq 0$$

$$\Leftrightarrow \langle T^{\{k+1+n\}}x, T^{\{k+1+n\}x}\rangle - (1+n)\langle T^{\{k+1\}}x, T^{\{k+1\}}x\rangle + n\langle T^k x, T^k x\rangle \geq 0$$

$$\Leftrightarrow T^{\{*k+1+n\}}T^{\{k+1+n\}} - (1+n)T^{\{*k+1\}}T^{\{k+1\}} + nT^{\{*k\}}T^k \geq 0$$

$$\Leftrightarrow \quad T^{\{*k\}}\left(T^{\{*1+n\}}T^{\{1+n\}} - (1+n)T^{\{*\}}T + nI\right)T^k \geq 0.$$

For example: let $x = (x_1, x_2, \dots) \in l^2$, Define $T: l^2 \to l^2$ by $T(x) = (0, x_1, x_2, \dots)$, $T^*(x) = (x_2, x_3, \dots)$. Then $T^{\{*k\}}\left(T^{\{*1+n\}}T^{\{1+n\}} - (1+n)T^{\{*\}}T + nI\right)T^k \geq 0$. ie T is k quasi n class Q operators or (n,k) quasi class Q operators.

From the definition of quasi n class Q operator we can easily say that every quasi n class Q operator is also an operator of k quasi n class Q. Hence we have the following implication

class $Q \subset n$ class $Q \subset$ quasi n class $Q \subset k$ quasi n class Q.

Theorem 2.3
Every k quasi class Q operator is (n,k) quasi class Q operator.
Proof
By using induction principle and simple calculation we get the result.

Corollary 2.4
If $T \in B(H)$ is of (n,k) quasi class Q then T is of $(n+1,k)$ quasi class Q operator

Corollary 2.5
If $T \in B(H)$ is of (n,k) quasi class Q then αT is (n,k) quasi class Q operator for any complex number α.

Theorem 2.6
Let $T \in B(H)$. If $\lambda^{\left\{\frac{-1}{2}\right\}}T$ is an operator of (n,k) quasi class Q, then T is k quasi n paranormal operator for all $\lambda > 0$.

Proof
Since $\lambda^{\left\{-\frac{1}{2}\right\}}T$ \$ is an operator of (n,k) quasi class Q, then

$$\left(\lambda^{-\frac{1}{2}}T\right)^{\{*k\}}\left(\left(\lambda^{-\frac{1}{2}}T\right)^{*(1+n)}\left(\lambda^{-\frac{1}{2}}T\right)^{\{1+n\}} - (1+n)\left(\lambda^{-\frac{1}{2}}T\right)^*\left(\lambda^{-\frac{1}{2}}T\right) + nI\right)\left(\lambda^{-\frac{1}{2}}T\right)^k \geq 0.$$

$$\left(\lambda^{-\frac{1}{2}}T\right)^{*(k+1+n)}\left(\lambda^{-\frac{1}{2}}T\right)^{k+1+n} - (1+n)\left(\lambda^{-\frac{1}{2}}T\right)^{*k+1}\left(\lambda^{-\frac{1}{2}}T\right)^{k+1} + n\left(\lambda^{-\frac{1}{2}}T\right)^{*k}\left(\lambda^{-\frac{1}{2}}T\right)^k \geq 0.$$

$$\left|\lambda^{-\frac{1}{2}}\right|^{2(k+1+n)}T^{*k+1+n}T^{k+1+n} - (1+n)\left|\lambda^{-\frac{1}{2}}\right|^{2(k+1)}T^{*(k+1)}T^{k+1} + n\left|\lambda^{-\frac{1}{2}}\right|^{2k}T^{*k}T^k \geq 0$$

By multiplying $|\lambda|^{k+1+n}$ and let $\lambda = \mu$, then

$$T^{*k+1+n}T^{k+1+n} - (1+n)\mu^n T^{*k+1}T^{(k+1)} + n\mu^{1+n}T^{*k}T^k \geq 0.$$

Hence T is k quasi n paranormal operator for all $\lambda > 0$.

Theorem 2.7
If (n,k) quasi class Q operator T doubly commutes with an isometric operator S, then TS is an operator of (n,k) quasi class Q.

Proof
Since T is (n,k) quasi class Q operator, then

$$T^{*k}\left(T^{*(1+n)}T^{1+n} - (1+n)T^*T + nI\right)T^k \geq 0.$$

Suppose T doubly commutes with an isometric operator S, then $TS = ST$, $S^*T = TS^*$ and $S^*S = I$. Now let $A = TS$. So we get $A^{*k}\left(A^{*(1+n)}A^{1+n} - (1+n)A^*A + nI\right)A^k \geq 0$. Therefore TS is a (n,k) quasi class Q operator.
Theorem 2.8

If a $(\square, k)$ quasi class Q operator $T \in B(H)$ is unitarily equivalent to operator S, then S is an operator of (n,k) quasi class Q.

Proof
Assume T is unitarily equivalent to operator S Then there exists an unitary operator U such that $S = U^*TU$ and T is (n,k) quasi class Q operator, then
$S^{*k}\left(S^{*(1+n)}S^{1+n} - (1+n)S^*S + nI\right)S^k$
$= (U^*TU)^{*k}\left((U^*TU)^{*(1+n)}(U^*TU)^{1+n} - (1+n)(U^*TU)^*(U^*TU) + nI\right)(U^*TU)^k \geq 0$. Therefore S is (n,k) quasi class Q operator.

Theorem 2.9
Let $T \in B(H)$ be an invertible operator and N be an operator such that N commutes with T^*T. Then operator N is (n,k) quasi class Q if and only if operator TNT^{-1} is of (n,k) quasi class Q.

Proof
Let N be (n,k) quasi class Q operator, then
$$N^{*k}\left(N^{*(1+\square)}N^{1+n} - (1+n)N^*N + nI\right)N^k \geq 0.$$
Since operator N commutes with operator T^*T, we have $(TNT^{-1})^{*k}\left((TNT^{-1})^{\{*1+n\}}(TNT^{-1})^{\{1+n\}} - (1+n)(TNT^{-1})^*(TNT^{-1}) + nI\right)(TNT^{-1})^{\{k\}}$
$$= T\left(N^{\{*k\}}\left(N^{\{*1+n\}}N^{\{1+n\}} - (1+n)N^*N + nI\right)N^k\right)T^{\{-1\}}.$$
Since N is (n,k) quasi class Q operator, then
$$T\left(N^{*k}\left(N^{\{*1+n\}}N^{\{1+n\}} - (1+n)N^*N + nI\right)N^k\right)T^* \geq 0.$$
Which implies (TT^*) commutes with $T\left(N^{*k}\left(N^{*(1+n)}N^{1+n} - (1+n)N^*N + nI\right)N^k\right)T^*$.
Also $(TT^*)^{-1}$ is also commutes with $T\left(N^{*k}\left(N^{*(1+n)}N^{1+n} - (1+n)N^*N + nI\right)N^k\right)T^*$.
Then $T\left(N^{*k}\left(N^{*(1+n)}N^{1+\square} - (1+n)N^*N + nI\right)N^k\right)T^{-1} \geq 0$.
Hence $TNT^{\{-1\}}$ is (n,k) quasi class Q operator.
Conversely suppose that $(TNT^{\{-1\}})$ is (n,k) quasi class Q operator, then
$$N^{\{*k\}}\left(N^{\{*1+n\}}N^{\{1+n\}} - (1+n)N^*N + nI\right)N^k \geq 0.$$

Corollary 2.10
Let S be (n,k) quasi class Q operator and A any positive operator such that $A^{-1} = A^*$. Then $T = A^{-1}SA$ is (n,k) quasi class Q operator.

Theorem 2.11
Let T be (n,k) quasi class Q operator. Then the tensor product $T \otimes I$ and $I \otimes T$ are both (n,k) quasi class Q operators.

Proof
By the definition of (n,k) quasi class Q and tensor product and by the simple calculation we get the result.

Theorem 2.12
If $T \in B(H)$ is of (n,k) quasi class Q operator for some positive integers k and n, the range of T does not have dense range then T has the following 2×2 matrix representation $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $H = \overline{ran(T^k)} \oplus ker\, T^{*k}$, if and only if T_1 is n class Q operator on $\overline{ran(T^k)}$ and $T_3^k = 0$. Further more $\sigma(T) = \sigma(T_1) \cup \{0\}$ where $\sigma(T)$ denotes the spectrum of T.

Proof
Let P be an orthogonal projection of H onto $\overline{ran(T^k)}$. Then $T_1 = TP = PTP$. By Theorem 2.2 we have that
$$T^{*k}\left(T^{*(1+n)}T^{1+n} - (1+n)T^*T + nI\right)T^k \geq 0$$
Which implies
$$P\left(T^{*1+n}T^{1+n} - (1+n)T^*T + nI\right)P \geq 0$$
Then $T_1^{*1+n}T_1^{1+n} - (1+n)T_1^*T_1 + nI \geq 0$
So T_1 is n-class Q operator on $\overline{ran(T^k)}$.
Also for any $x = (x_1, x_2) \in H$,
$$\langle T_3^k x_2, x_2\rangle = \langle T^k(I-P)x, (I-P)x\rangle$$
$$= \langle (I-P)x, T^{*k}(I-P)x\rangle = 0$$
This implies $T_3^k = 0$.
Since $\sigma(T) \cup \tau = \sigma(T_1) \cup \sigma(T_3)$ where τ is the union of certain holes in $\sigma(T)$, which happens to be a subset of $\sigma(T_1) \cap \sigma(T_3)$ [by corollary 7, [9]] and $\sigma(T_3) = 0$. $\sigma(T_1) \cap \sigma(T_3)$ has no interior points. So we have $\sigma(T) = \sigma(T_1) \cup \{0\}$.
Suppose that $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on H = $\overline{ran(T^k)} \oplus \backslash ker\, T^{*k}$ where T_1 is n class Q operator on $\overline{ran(T^k)}$ and $T_3^k = 0$. Then
$$T^k = \begin{pmatrix} T_1^k & \sum_{j=0}^{k-1} T_1^j T_2 T_3^{k-1-j} \\ 0 & 0 \end{pmatrix}$$

$$T^{*k} = \begin{pmatrix} T_1^{*k} & 0 \\ \left(\sum_{j=0}^{k-1} T_1^j T_2 T_3^{k-1-j}\right)^* & 0 \end{pmatrix}$$

$$T^{*k}\left(T^{*(1+n)}T^{\{1+n\}} - (1+n)T^*T + nI\right)T^k$$
$$= \begin{pmatrix} T_1^{*k}\left(T_1^{\{*1+n\}}T_1^{\{1+n\}} - (1+n)T_1^*T_1 + nI\right)T_1^k & X \\ X^* & Y \end{pmatrix}$$

Where $X = T_1^{*k}(T_1^{\{*(1+n)\}}T_1^{1+n} - (1+n)T_1^*T_1 + nI)(\sum_{\{j=0\}}^{\{k-1\}} T_1^j T_2 T_3^{k-1-j})$

$$Y = \left(\sum_{j=0}^{\{k-1\}} T_1^j T_2 T_3^{\{k-1-j\}}\right)^* (T_1^{\{*1+n\}}T_{\{1\}}^{\{1+n\}} - (1+n)T_{\{1\}}^*T_{\{1\}} + nI)\ (\sum_{\{j=0\}}^{\{k-1\}} T_1^j T_2 T_3^{\{k-1-j\}})$$

We know that, " If A is a matrix of the form $\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \geq 0$ if and only if $A \geq 0,\ C \geq 0$ and $B = A^{\frac{1}{2}}WC^{\frac{1}{2}}$ \$ for some contraction W. Since T_1 is n-class Q operator and $Y \geq 0$, then we have $T^{*k}\ (T^{*1+n}T^{1+n} - (1+n)T^*T + nI)T^k \geq 0$. Hence T is (n,k) quasi class Q operator.

Theorem 2.13.
Let M be a closed T -invariant subspace of H. Then the restriction $T|_M$ of is (n,k) quasi class Q operator T to M is (n,k) quasi class Q operator.

Proof
Let $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $H = M \oplus M^{\perp}$. Since T is (n,k) quasi class Q operator then by Theorem 2.12, we have $T|_M$ is also is (n,k) quasi class Q operator.

Theorem 2.14
Let T be a regular is (n,k) quasi class Q operator, then the approximate point spectrum lies in the disc

$$\sigma_{ap}(T) \subseteq \{\lambda \in C\colon \frac{(1+n)^{\frac{1}{2}}}{\|T^{-(k+1)}\|(\|T^{1+n}\|^2+n)^{\frac{1}{2}}} \leq |\lambda| \leq \|T\|$$

Proof
Suppose T is regular (n,k) quasi class Q operator, then for every unit vector x in H, we have

$$\|x\|^2 \leq \ \left\|T^{-(k+1)}\right\|^2\left\|T^{(k+1)x}\right\|^2 \ \leq \frac{\left\|T^{-(k+1)}\right\|^2}{1+n}(\|T^{1+n}\|^2\|T^kx\|^2 \ + n\ \|T^kx\|^2).$$

$$\text{Hence } \|T^kx\|^2 \geq \frac{(1+n)\|x\|^2}{\left\|T^{-(k+1)}\right\|^2(\|T^{1+n}\|^2+n)}\ .$$

Now assume that $\lambda \in \sigma_{ap}(T)$. Then there exists a sequence $\{\, x_m\},\ \ \|x_m\| = 1$ such that

$\|(T-\lambda)x_m\| \to 0$ when $m \to \infty$. So we have $\|Tx_m - \lambda x_m\| \geq \|Tx_m\| - |\lambda|\|x_m\| \geq \frac{(1+n)^{\frac{1}{2}}}{\|T^{-(k+1)}\|(\|T^{1+n}\|^2+n)^{\frac{1}{2}}} - |\lambda|$. Now,

when $m \to \infty$, $|\lambda| \geq \frac{(1+n)^{\frac{1}{2}}}{\|T^{-(k+1)}\|(\|T^{1+n}\|^2+n)^{\frac{1}{2}}}$.

III. (n,k) Quasi class Q^* operators

In this section we define operators of (n,k) quasi class Q and consider some basic properties and examples.

Definition 3.1
An operator T is said to be (n,k) quasi class Q^* (quasi n-class Q^*)if

$$\|T^*T^kx\|^2 \leq \frac{1}{1+n}\ (\|T^{k+1+n}x\|^2 + n\|T^kx\|^2)$$

for every $x \in H$ and every positive integer n. When $n = 1$, it is of k quasi class Q^* (k quasi *-class Q)operator and when $k = 1$, it is of quasi n class Q^* operator.

For example: let $x = (x_1, x_2, \dots) \in l^2$, Define $T\colon l^2 \to l^2$ by $T(x) = (0, x_1, x_2, \dots)$, $T^*(x) = (x_2, x_3, \dots)$. Then $T^{*k}(T^{*1+n}T^{1+n} - (1+n)TT^* + nI)T^k \geq 0$. ie T is (n,k) quasi class Q^* operator.

Using the definition of (n,k) quasi class Q^* operator and by simple calculation we get the following theorem.

Theorem 3.2
For each positive integer n, T is of (n,k) quasi class Q^* operator if and only if

$$T^{*k}(T^{*1+n}T^{1+n} - (1+n)TT^* + nI)T^k \geq 0.$$

From the definition of (n,k) quasi class Q^* operator, we can easily say that every operator of n-class $Q\wedge *$ and quasi n-class Q^* is also an operator of (n,k) quasi class Q^*. Hence we have the following implications

class $Q^* \subset$ n class $Q^* \subset$ quasi n-class $Q^* \subset$ k \$ quasi n class Q^*

Also every (n,k) quasi class Q^* is $(n+1,k)$ quasi class Q^* operator. Again, if $T \in B(H)$ is (n,k) quasi class Q^* then αT is of (n,k) quasi class Q^* operator for any complex number α.

Theorem 3.3

Let $T \in B(H)$. If $\lambda^{-\frac{1}{2}}T$ is an operator of (n,k) quasi class Q^*, then T is k quasi *- n -paranormal operator for all $\lambda > 0$.

Theorem 3.4

If (n,k) quasi class Q^* operator T doubly commutes with an isometric operator S, then TS is an operator of (n,k) quasi class Q^*.

Theorem 3.5

If (n,k) quasi class Q^* operator $T \in B(H)$ is unitarily equivalent to operator S, then S is an operator of (n,k) quasi class Q^*.

Theorem 3.6

Let $T \in B(H)$ be an invertible operator and N be an operator such that N commutes with T^*T. Then operator N is (n,k) quasi class Q^* if and only if operator TNT^{-1} is (n,k) quasi class Q^*.

Corollary 3.7

Let S be (n,k) quasi class Q^* operator and A any positive operator such that $A^{-1} = A^*$. Then $T = A^{-1}SA$ is (n,k) quasi class Q^* operator.

Thereom 3.8

Let T be (n,k) quasi class Q^* operator. Then the tensor product $T \otimes I$ and $I \otimes T$ are both (n,k) quasi class Q^* operators.

Theorem 3.9

If $T \in B(H)$ is of (n,k) quasi class Q^* operator for any positive integer n, a non zero complex number $\lambda \in \sigma_p(T)$ and T is of the form $T = \begin{pmatrix} \lambda & T_2 \\ 0 & T_3 \end{pmatrix}$ on $H = ker(T-\lambda) \oplus \overline{ran(T-\lambda)^*}$, then

1. $T_2 = 0$ and
2. T_3 is (n,k) quasi class Q^*operator.

Proof

Let $T = \begin{pmatrix} \lambda & T_2 \\ 0 & T_3 \end{pmatrix}$ on H = $\overline{\ker(T-\lambda)} \oplus \overline{ran(T-\lambda)^*}$ Without the loss of generality assume that $\lambda = 1$, then by Theorem 3.2, $T^{*k}(T^{*1+n}T^{1+n} - (1+n)TT^* + nI)T^k \geq 0$. Then,

$$T^k = \begin{pmatrix} 1 & \sum_{j=0}^{k-1} T_2\, T_3^{k-1-j} \\ 0 & T_3^k \end{pmatrix} \text{ and}$$

$$T^{*\square} = \begin{pmatrix} 1 & 0 \\ \left(\sum_{j=0}^{k-1} T_2\, T_3^{k-1-j}\right)^* & T_3^{*k} \end{pmatrix}$$

So, $T^{*k}\,(T^{*1+n}T^{1+n} - (1+n)TT^* + nI)T^k \geq 0$ gives

$$\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \geq 0$$

Where

$A = 1 - 1(1+n)(1+T_2T_2^*) + n$,

$B = \left(\sum_{j=0}^{n} T_2\, T_3^{n-j} - (1+n)T_2T_3^*\right)T_3^k - (1+n)(T_2T_2^*)\left(\sum_{j=0}^{k-1} T_2\, T_3^{k-1-j}\right)$ and

$$C = B^*\left(\sum_{j=0}^{k-1} T_2\, T_3^{k-1-j}\right) + \left(\sum_{j=0}^{k-1} T_2\, T_3^{k-1-j}\right)^*\left(\sum_{j=0}^{n} T_2\, T_3^{n-j} - (1+n)T_2T_3^*\right)$$
$$+\, T_3^k\left(\left(\sum_{j=0}^{n} T_2\, T_3^{n-j}\right)^*\left(\sum_{j=0}^{n} T_2\, T_3^{n-j}\right)\right) + T_3^{*k}(T_3^{*1+n}T_3^{1+n} - (1+n)T_3T_3^* + nI)T_3^k$$

Therefore $1 + n - (1+n)(1+T_2T_2^*) + n \geq 0$, which implies that $(1+n)(-T_2T_2^*) \geq 0$. This gives $T_2 = 0$, since n is a positive integer. Hence T_3 is k quasi n-class Q^* operator.

Corollary 3.10

If $T \in B(H)$ is of (n,k) quasi class Q^* operator for a positive integer n, then T is of the form $T = \begin{pmatrix} \lambda & 0 \\ 0 & T_3 \end{pmatrix}$ on $H = ker\,(T-\lambda) \oplus \overline{\{ran(T-\lambda)\}}^*$, where T_3 is (n,k) quasi class Q^* operator and $ker\,(T-\lambda) = \{0\}$.

Theorem 3.11

If $T \in B(H)$ is (n,k) quasi class Q^* operator for a positive integer n, T does not have dense range and T has the following 2×2 matrix representation $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $H = \overline{ran(T^k)} \oplus ker\, T^{*k}$ if and only if $T_1^{*1+n}T_1^{1+n} - (1+n)(T_1T_1^* + T_2T_2^*) + nI \geq 0$ \$ and $T_3^k = 0$. Further more $\sigma(T) = \sigma(T_1) \cup \{\,0\,\}$.

Proof

Let $T \in B(H)$ be k quasi n class Q^* operator and P be an orthogonal projection onto $ran(T^k)$. Then $T_1 = TP = PTP$. By Theorem 3.2 we have that

$$T^{*k}(T^{*1+n}T^{1+n} - (1+n)TT^* + nI)T^k \geq 0$$
$$P(T^{*1+n}T^{1+n} - (1+n)TT^* + nI)P \geq 0$$
$$T_1^{*1+n}T_1^{1+n} - (1+n)(T_1T_1^* + T_2T_2^*) + nI \geq 0$$

Also for any $x = (x_1;\, x_2) \in H$,

$$\langle T_3^k x_2, x_2\rangle = \langle T^k(I-P)x, (I-P)x\rangle$$
$$= \langle (I-P)x, T^{*k}(I-P)x\rangle = 0$$

This implies $T_3^k = 0$.

Since $\sigma(T) \cup \tau = \sigma(T_1) \cup \sigma(T_3)$ where τ is the union of certain holes in $\sigma(T)$, which happens to be a subset of $\sigma(T_1) \cap \sigma(T_3)$ [by corollary 7, [9]]. $\sigma(T_3) = 0$ and $\sigma(T_1) \cap \sigma(T_3)$ has no interior points we have $\sigma(T) = \sigma(T_1) \cup \{0\}$.

Suppose that $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $H = \overline{ran(T^k)} \oplus \ker T^{*k}$, $T_1^{*1+n}T_1^{1+n} - (1+n)$ $(T_1T_1^* + T_2T_2^*) + nI \geq 0$ and $T_3^k = 0$. Then we have

$T^{*k}\,(T^{*1+n}T^{1+n} - (1+n)TT^* + nI)T^k$

$$= \begin{pmatrix} T_1^{*k} & 0 \\ \left(\sum_{j=0}^{k-1} T_1^j T_2\, T_3^{k-1-j}\right)^* & T_3^{*k} \end{pmatrix}$$
$$\begin{pmatrix} T_1^{*1+n}T_1^{1+n} - (1+n)(T_1T_1^* + T_2T_2^*) + n & T_1^{*1+n}\left(\sum_{j=0}^{n} T_1^j T_2\, T_3^{n-j}\right) - (1+n)T_2T_3^* \\ \left(\sum_{j=0}^{n} T_1^j T_2\, T_3^{n-j}\right)^* T_1^{1+n} - (1+n)T_3T_2^* & \begin{bmatrix} \left(\sum_{j=0}^{n} T_1^j T_2\, T_3^{n-j}\right)^*\left(\sum_{j=0}^{n} T_1^j T_2\, T_3^{n-j}\right) \\ -(1+n)T_3T_3^* + n \end{bmatrix} \end{pmatrix}$$
$$\begin{pmatrix} T_1^k & \sum_{j=0}^{k-1} T_1^j T_2\, T_3^{k-1-j} \\ 0 & T_3^k \end{pmatrix}$$
$$= \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \geq 0.$$

Where

$A = T_1^{*k}(T_1^{*1+n}T_1^{1+n} - (1+n)(T_1T_1^* + T_2T_2^*) + nI)T_1^k$,

$B = T_1^{*k}(T_1^{*1+n}T_1^{1+n} - (1+n)(T_1T_1^* + T_2T_2^*) + nI)\left(\sum_{j=0}^{k-1} T_1^j T_2\, T_3^{k-1-j}\right)$ and

$C = \left(\sum_{j=0}^{k-1} T_1^j T_2 T_3^{k-1-j}\right)^* (T_1^{*1+n}T_1^{1+n} - (1+n)(T_1T_1^* + T_2T_2^*) + nI)\left(\sum_{j=0}^{k-1} T_1^j T_2 T_3^{k-1-j}\right)$

Hence T is k quasi n-class Q^* operator.

Theorem 3.12

Let M be a closed T -invariant subspace of H. Then the restriction $T|_M$ of is (n,k) quasi class Q^* operator T to M is (n,k) quasi class Q^* operator.

Proof

By Theorem 3.11, $T|_M$ is also k quasi n class Q^* operator.

Theorem 3.13

Let T be a regular is (n,k) quasi class Q^* operator, then the approximate point spectrum lies in the disc $\sigma_{ap}(T) \subseteq \{\lambda \in C: \frac{(1+n)^{\frac{1}{2}}}{\|T^{-(k)}\|\|T^{*-1}\|(\|T^{1+n}\|^2+n)^{\frac{1}{2}}} \leq |\lambda| \leq \|T\|$

Proof

Suppose T is regular k quasi n class Q^* operator, then for every unit vector x in H, we have

$$\|T^k x\|^2 \geq \frac{(1+n)\|x\|^2}{\|T^{-k}\|^2\|T^{*-1}\|(\|T^{1+n}\|^2 + n)}$$

Now assume that $\lambda \in \sigma_{ap}(T)$. Then there exists a sequence $\{x_m\}\|x_m\| = 1$ such that $\|(T-\lambda)x_m\| \to 0$ when $m \to \infty$ we have

$$\|Tx_m - \lambda x_m\| \geq \|Tx_m\| - |\lambda|\|x_m\|$$
$$\geq \|T\| - |\lambda|$$
$$\geq \frac{(1+n)^{\frac{1}{2}}}{\|T^{*-1}\|\|T^{-k}\|(\|T^{1+n}\|^2 + n)^{\frac{1}{2}}} - |\lambda|$$

Now when $m \to \infty$, $|\lambda| \geq \frac{(1+n)^{\frac{1}{2}}}{\|T^{*-1}\|\|T^{-k}\|(\|T^{1+n}\|^2+n)^{\frac{1}{2}}}$.

IV. (n,k) QUASI CLASS Q AND (n,k) QUASI CLASS Q^* COMPOSITION OPERATORS

Let $L^2(\lambda) = L^2(X,\Sigma,\lambda)$, where (X,Σ,λ) be a sigma-finite measure space. A bounded linear operator $C_T f = f \circ T$ on $L^2(X,\Sigma,\lambda)$ is said to be a composition operator induced by T, a non-singular measurable transformation from X into itself, when the measure λT^{-1} is absolutely continuous with respect to the measure λ and the Radon-Nikodym derivative $\frac{d\lambda T^{-1}}{d\lambda} = f_0$ is essentially bounded. The Radon-Nikodym derivative of the measure $\lambda(T^k)^{-1}$ with respect to λ is denoted by $f_0^{(k)}$, where T^k is obtained by composing T- k times. Every essentially bounded complex-valued measurable function f_0 induces the bounded operator M_{f_0} on $L^2(\lambda)$, which is defined by $M_{f_0} f = f_0 f$ for every $f \in L^2(\lambda)$. Further $C_T^* C_T = M_{\{f_0\}}$, $C_T^{*2} C_T^2 = M_{f_0^{(2)}}$ and $C_T^{*1+n} C_T^{1+n} = M_{f_0^{1+n}}$.

The following lemma due to Harrington and Whitley [9] is well known.

Lemma 4.1

Let P denote the projection of L^2 on $\overline{R(C)}$

(1) $C_T^* C_T f = f_0 f$ and $C_T C_T^* f = (f_0 \circ T)Pf$ for all $f \in L^2$, where P is the projection of L^2 onto $\overline{R(C)}$.

(2) $\overline{R(C)} = \{f \in L^2 : f \text{ is } T^{-1}\Sigma \text{ measurable}\}$.

In this section k quasi n-class Q and k quasi n-class Q^* composition operator on L^2 space are characterized as follows.

Theorem 4.2.

Let $C_T \in B(L^2(\lambda))$. Then C_T is of k quasi n-class Q if and only if $f_0^{(k+1+n)} - (1+n)f_0^{(k+1)} + nf_0^{(k)} \geq 0$ a.e.

Proof

Let $C_T \in B(L^2(\lambda))$ is of k quasi n-class Q if and only if

$$C_T^{*k+1+n} C_T^{k+1+n} - (1+n)C_T^{*k+1} C_T^{k+1} + nC_T^{*k} C_T^k \geq 0$$

By Theorem 2.2

Thus $\langle (C_T^{*k+1+n} C_T^{k+1+n} - (1+n)C_T^{*k+1} C_T^{k+1} + nC_T^{*k} C_T^k)\chi_E, \chi_E \rangle \geq 0$ for every characteristic function χ_E of E in Σ such that $\lambda(E) < \infty$. Since $C_T^* C_T = M_{f_0}$ and $C_T^{*k+1+n} C_T^{k+1+n} = M_{f_0^{(k+1+n)}}$ then $\langle \left(M_{f_0^{(k+1+n)}} - (1+n)M_{f_0^{(k+1)}} + nM_{f_0^{(k)}}\right)\chi_E, \chi_E \rangle \geq 0$.

Hence $\int_E \left(f_0^{(k+1+n)} - (1+n)f_0^{(k+1)} + nf_0^{(k)}\right) d\lambda \geq 0$ for every E in Σ.

Hence C_T is of k quasi n-class Q if and only if $f_0^{(k+1+n)} - (1+n)f_0^{(k+1)} + nf_0^{(k)} \geq 0$ a.e.

Example 4.3

Let $X = N$, the set of all natural numbers and λ be the counting measure on it. Define $T: N \to N$ by $T(1) = 1, T(4p+q-2) = p+1$ for $q = 0,1,2,3$ and $p \in N$. We have $f_0^p = f_0^2(p) = \cdots = f_0^n(p) = 1$ for $p = 1$. $f_0(p) = 4$, $f_0^2(p) = 16, \ldots = f_0^{(k+1+n)}(p) = 4^{k+1+n}$ for $p \in N - \{1\}$. Since $f_0^{(k+1+n)}(p) - (1+n)f_0^{(k+1)}(p) + nf_0^{(k)}(p) \geq 0$ for every p. Hence Hence C_T is of k quasi n-class Q operator.

Theorem 4.4 [17]

If $C_T \in B(L^2(\lambda))$ has dense range then $f_0 = g_0 \circ T$ a.e.

Corollary 4.5

If C_T is of k quasi n-class Q with dense range on $L^2(\lambda)$ then $(g_0 \circ T)^{(k+1+n)} - (1+n)(g_0 \circ T)^{(k+1)} + n(g_0 \circ T)^k \geq 0$ a.e.

Proof

By Theorem 4.2 and Theorem 4.4, we obtain the result.

Theorem 4.6

Let $C_T \in B(L^2(\lambda))$. Then C_T^* is of k quasi n-class Q operator if and only if $(f_0^{k+1+n} \circ T^{k+1+n})P_{k+1+n} - (1+n)(f_0^{k+1} \circ T^{k+1})P_{k+1} + n(f_0^k \circ T^k)P_k \geq 0$ a.e, where P_1, P_2, P_{k+1+n} are the projections of L^2 onto $\overline{R(C)}, \overline{R(C^2)}, \ldots, \overline{R(C^{k+1+n})}$ respectively.

Proof

Suppose $C_T \in B(L^2(\lambda))$ and C_T^* is of k quasi n-class Q operator if and only if

$$C_T^{k+1+n} C_T^{*k+1+n} - (1+n)C_T^{k+1} C_T^{*k+1} + nC_T^k C_T^{*k} \geq 0$$

By Theorem 2.2. Then

$$\langle (C_T^{k+1+n} C_T^{*k+1+n} - (1+n)C_T^{k+1} C_T^{*k+1} + nC_T^k C_T^{*k})f, f \rangle \geq 0 \text{ for every } f \in L^2(\lambda).$$

Since $\langle C_T C_T^* f, f \rangle = \langle (f_0 \circ T)P_1 f, f \rangle$By [10]. Hence $\langle (f_0^{k+1+n} \circ T^{k+1+n})P_{k+1+n} f, f \rangle - (1+n)\langle (f_0^{k+1} \circ T^{k+1})P_{k+1} f, f \rangle + n\langle (f_0^k \circ T^k)P_k f, f \rangle \geq 0$ for every $f \in L^2(\lambda)$.

Hence

$$\langle\left((f_0^{k+1+n} \circ T^{k+1+n})P_{k+1+n} - (1+n)(f_0^{k+1} \circ T^{k+1})P_{k+1} + n(f_0^k \circ T^k)P_k\right) f, f\rangle \geq 0,$$
$$\Leftrightarrow \quad (f_0^{k+1+n} \circ T^{k+1+n})P_{k+1+n} - (1+n)(f_0^{k+1} \circ T^{k+1})P_{k+1} + n(f_0^k \circ T^k)P_k \geq 0 \text{a.e.}$$

Corollary 4.7

Let $C_T \in B(L^2(\lambda))$ with dense range. Then C_T^* is of k quasi n-class Q operator if and only if $(f_0^{k+1+n} \circ T^{k+1+n}) - (1+n)(f_0^{k+1} \circ T^{k+1}) + n(f_0^k \circ T^k) \geq 0$ a.e.

Theorem 4.8

Let $C_T \in B(L^2(\lambda))$. Then C_T is of k quasi n-class Q^* if and only if $f_0^{(k+1+n)} - (1+n)f_0^{(k)}E(f_0) \circ T^{-k} + nf_0^{(k)} \geq 0$ a.e.

Proof

Let $C_T \in B(L^2(\lambda))$ is of k quasi n-class Q^* if and only if

$$C_T^{*k+1+n}C_T^{k+1+n} - (1+n)C_T^{*k}(C_TC_T^*)C_T^k + nC_T^{*k}C_T^k \geq 0$$

Thus $\langle(C_T^{*k+1+n}C_T^{k+1+n} - (1+n)C_T^{*k}(C_TC_T^*)C_T^k + nC_T^{*k}C_T^k)\chi_E, \chi_E\rangle \geq 0$ for every characteristic function χ_E of E in Σ such that $\lambda(E) < \infty$. Since $C_T^*C_T = M_{f_0}$ and $C_T^{*k+1+n}C_T^{k+1+n} = M_{f_0^{(k+1+n)}}$ then $\int_E\left(f_0^{(k+1+n)} - (1+n)f_0^{(k)}E(f_0) \circ T^{-k} + nf_0^{(k)}\right) d\lambda \geq 0$ for every E in Σ.

Hence C_T is of k quasi n-class Q^* if and only if $f_0^{(k+1+n)} - (1+n)f_0^{(k)}E(f_0) \circ T^{-k} + nf_0^{(k)} \geq 0$ a.e.

Example 4.9

Let $X = N$, the set of all natural numbers and λ be the counting measure on it. Define $T: N \to N$ by $T(1) = T(2) = T(3) = 1$, $T(4p+q) = p+1$ for $q = 0,1,2,3$ and $p \in N$. Since $f_0^{(k+1+n)} - (1+n)f_0^{(k)}E(f_0) \circ T^{-k} + nf_0^{(k)} \geq 0$ for every p. Hence Hence C_T is of k quasi n-class Q^* operator.

Corollary 4.10

If C_T is k quasi n-class Q^* with dense range on $L^2(\lambda)$ if and only if $f_0^{k+1+n} - (1+n)f_0^{k+1} + nf_0^k \geq 0$ a.e.

Theorem 4.11

Let $C_T \in B(L^2(\lambda))$. Then C_T^* is of k quasi n-class Q operator if and only if $(f_0^{k+1+n} \circ T^{k+1+n})P_{k+1+n} - (1+n)(f_0^{k+1} \circ T^{k+1})P_{k+1} + n(f_0^k \circ T^k)P_k \geq 0$ a.e, where P_i's are the projections of L^2 onto $\overline{R(C^i)}$ respectively.

Proof

Let $C_T^* \in B(L^2(\lambda))$ is of k quasi n-class Q operator if and only if

$$C_T^{k+1+n}C_T^{*k+1+n} - (1+n)C_T^k(C_T^*C_T)C_T^{*k} + nC_T^kC_T^{*k} \geq 0$$

Thus

$\langle(C_T^{k+1+n}C_T^{*k+1+n} - (1+n)C_T^k(C_T^*C_T)C_T^{*k} + nC_T^kC_T^{*k})f, f\rangle \geq 0$ for every characteristic function χ_E of E in Σ such that $\lambda(E) < \infty$. Since $C_T^*C_T = M_{f_0}$, $C_T^{*1+n}C_T^{1+n} = M_{f_0^{(1+n)}}$ and $C_TC_T^* = (f_0 \circ T)P$ then $\int_E\left((f_0^{k+1+n} \circ T^{k+1+n})P_{k+1+n} - (1+n)f_0^k(f_0 \circ T^{k-1}) + n(f_0^k \circ T^k)P_k\right) d\lambda \geq 0$ for every E in Σ.

Hence C_T is of k quasi n-class Q^* if and only if $(f_0^{k+1+n} \circ T^{k+1+n})P_{k+1+n} - (1+n)f_0^k(f_0 \circ T^{k-1}) + n(f_0^k \circ T^k)P_k \geq 0$ a.e.

Corollary 4.12

Let $C_T \in B(L^2(\lambda))$ with dense range. Then C_T^* is of k quasi n-class Q^* if and only if $(f_0^{k+1+n} \circ T^{k+1+n}) - (1+n)f_0^k(f_0 \circ T^{k-1}) + n(f_0^k \circ T^k) \geq 0$ a.e.

V. k Quasi n-class Q and k Quasi n-class Q^* Weighted Composition Operators

A weighted composition operator is a linear transformation acting on the set of complex valued Σ measurable functions f of the form $W_Tf = w(f \circ T)$, where w is a complex valued measurable function. In the case that $w = 1$ a.e., we say that W_T is a composition operator. Let w_k denote $w(w_T)(w_T^2), \ldots, (w_T^{k-1})$ so that $W_k^Tf = w_k(f \circ T)^k$ [12].

To examine the weighted composition operators efficiently, Alan Lambert [11], associated conditional expectation operator E with each transformation T as $E(\bullet|T^1\Sigma) = E(\bullet)$.

$E(f)$ is defined for each non-negative measurable function $f \in L^p(1 \leq p)$ and is uniquely determined by the conditions

(i) $E(f)$ is $T^{-1}\Sigma$ measurable and

(ii) If B is any $T^1\Sigma$ measurable set for which $\int_B f\, d_\lambda$ converges, then we have $\int_B f\, d_\lambda = \int_B E(f)\, d_\lambda$.

As an operator on L^p, E is the projection onto the closure range of C. E_n the identity on L^p if and only if $T^{-1}\sigma = \sigma$. Now we are ready to derive the characterization of k quasi n-class Q and of k quasi n-class Q^* weighted composition operator as follows.

Theorem 5.1

Let W_T be a weighted composition operator on $B(L^2(\lambda))$. Then W_T is of k quasi n-class Q if and only if $\left(f_0^{k+1+n}E(w_{k+1+n}^2) \circ T^{-(k+1+n)}\right) - (1+n)\left(f_0^{(k+1)}E(w_{k+1}^2) \circ T^{-(k+1)}\right) + n(f_0^kE(w_k^2) \circ T^{-k}) \geq 0$ a.e.

Proof

Let $W_T \in B(L^2(\lambda))$ is of k quasi n-class Q if and only if

$$W_T^{*k+1+n}W_T^{k+1+n} - (1+n)W_T^{*k+1}W_T^{k+1} + nW_T^{*k}W_T^{k} \geq 0$$

By Theorem 2.2

Thus $\langle (W_T^{*k+1+n}W_T^{k+1+n} - (1+n)W_T^{*k+1}W_T^{k+1} + nW_T^{*k}W_T^{k})\chi_E, \chi_E\rangle \geq 0$ for every characteristic function χ_E of E in Σ such that $\lambda(E) < \infty$. Since $W_T^*W_T = f_0E(w^2)\circ T^{-1}$, $W_T^kf = w_k(f\circ T)^k$, $W_T^{*k}f = f_0^kE(w_kf)\circ T^{-k}$ and $W_T^{*k}W_T^kf = f_0^kE(w_k^2)\circ T^{-k}f$. Then $\langle \left(f_0^{k+1+n}E(w_{k+1+n}^2)\circ T^{-(k+1+n)} - (1+n)\left(f_0^{(k+1)}E(w_{k+1}^2)\circ T^{-(k+1)}\right) + n(f_0^kE(w_k^2)\circ T^{-k})\right)\chi_E, \chi_E\rangle \geq 0$.

Which implies $\int_E \left(f_0^{k+1+n}E(w_{k+1+n}^2)\circ T^{-(k+1+n)}\right) - (1+n)\left(f_0^{(k+1)}E(w_{k+1}^2)\circ T^{-(k+1)}\right) + n(f_0^kE(w_k^2)\circ T^{-k})d\lambda \geq 0$ for every E in Σ.

Hence W_T is of k quasi n-class Q if and only if $\left(f_0^{k+1+n}E(w_{k+1+n}^2)\circ T^{-(k+1+n)}\right) - (1+n)\left(f_0^{(k+1)}E(w_{k+1}^2)\circ T^{-(k+1)}\right) + n(f_0^kE(w_k^2)\circ T^{-k}) \geq 0$ a.e.

Corollary 5.2

Let W_T be a weighted composition operator on $B(L^2(\lambda))$ and assume that $T^{-1}\Sigma = \Sigma$. Then W_T is of k quasi n-class Q if and only if $\left(f_0^{k+1+n}(w_{k+1+n}^2)\circ T^{-(k+1+n)}\right) - (1+n)\left(f_0^{(k+1)}(w_{k+1}^2)\circ T^{-(k+1)}\right) + n(f_0^k(w_k^2)\circ T^{-k}) \geq 0$ a.e.

Theorem 5.3

Let W_T be a weighted composition operator on $B(L^2(\lambda))$. Then W_T^* is of k quasi n-class Q if and only if $w_{k+1+n}\left(f_0^{k+1+n}\circ T^{-(k+1+n)}\right)E(w_{k+1+n}) - (1+n)w_{k+1}\left(f_0^{(k+1)}\circ T^{-(k+1)}\right)E(w_{k+1}) + nw_k(f_0^k\circ T^{-k})E(w_k) \geq 0$ a.e.

Proof

Let $W_T^* \in B(L^2(\lambda))$ is of k quasi n-class Q if and only if

$$W_T^{k+1+n}W_T^{*k+1+n} - (1+n)W_T^{k+1}W_T^{*k+1} + nW_T^{k}W_T^{*k} \geq 0$$

By Theorem 2.2

Thus $\langle (W_T^{k+1+n}W_T^{*k+1+n} - (1+n)W_T^{k+1}W_T^{*k+1} + nW_T^{k}W_T^{*k})\chi_E, \chi_E\rangle \geq 0$ for every characteristic function χ_E of E in Σ such that $\lambda(E) < \infty$. Since $W_TW_T^* = w(f_0\circ T)E(w\, f)$, $W_T^kf = w_k(f\circ T)^k$, $W_T^{*k}f = f_0^kE(w_kf)\circ T^{-k}$ and $W_T^{*k}W_T^kf = w_k(f_0^k\circ T^{\,k})E(w_kf)$. Then $\langle \left(w_{k+1+n}\left(f_0^{k+1+n}\circ T^{-(k+1+n)}\right)E(w_{k+1+n}) - (1+n)w_{k+1}\left(f_0^{(k+1)}\circ T^{-(k+1)}\right)E(w_{k+1}) + nw_k(f_0^k\circ T^{-k})E(w_k)\right)\chi_E, \chi_E\rangle \geq 0$.

Which implies $\int_E w_{k+1+n}\left(f_0^{k+1+n}\circ T^{-(k+1+n)}\right)E(w_{k+1+n}) - (1+n)w_{k+1}\left(f_0^{(k+1)}\circ T^{-(k+1)}\right)E(w_{k+1}) + nw_k(f_0^k\circ T^{-k})E(w_k)d\lambda \geq 0$ for every E in Σ.

Hence W_T^* is of k quasi n-class Q if and only if $w_{k+1+n}\left(f_0^{k+1+n}\circ T^{-(k+1+n)}\right)E(w_{k+1+n}) - (1+n)w_{k+1}\left(f_0^{(k+1)}\circ T^{-(k+1)}\right)E(w_{k+1}) + nw_k(f_0^k\circ T^{-k})E(w_k) \geq 0$ a.e.

Corollary 5.4

Let W_T be a weighted composition operator on $B(L^2(\lambda))$ and $T^{-1}\Sigma = \Sigma$. Then W_T^* is of k quasi n-class Q if and only if $w_{k+1+n}^2\left(f_0^{k+1+n}\circ T^{-(k+1+n)}\right) - (1+n)w_{k+1}^2\left(f_0^{(k+1)}\circ T^{-(k+1)}\right) + nw_k^2(f_0^k\circ T^{-k}) \geq 0$ a.e.

Theorem 5.5

Let W_T be a weighted composition operator on $B(L^2(\lambda))$. Then W_T is of k quasi n-class Q^* if and only if $\left(f_0^{k+1+n}E(w_{k+1+n}^2)\circ T^{-(k+1+n)}\right) - (1+n)\left(f_0^{(k)}E(w_{k+1}^2)E(f_0)\circ T^{-k}\right) + n(f_0^kE(w_k^2)\circ T^{-k}) \geq 0$ a.e.

Proof

Let $W_T \in B(L^2(\lambda))$ is of k quasi n-class Q if and only if

$$W_T^{*k+1+n}W_T^{k+1+n} - (1+n)W_T^{*k}W_TW_T^*W_T^k + nW_T^{*k}W_T^k \geq 0$$

By Theorem 2.2

Thus $\langle (W_T^{*k+1+n}W_T^{k+1+n} - (1+n)W_T^{*k}W_TW_T^*W_T^k + nW_T^{*k}W_T^k)\chi_E, \chi_E\rangle \geq 0$ for every characteristic function χ_E of E in Σ such that $\lambda(E) < \infty$. Since $W_T^*W_T = f_0E(w^2)\circ T^{-1}f$, $W_T^kf = w_k(f\circ T)^k$, $W_T^{*k}f = f_0^kE(w_kf)\circ T^{-k}$ and $W_T^{*k}W_T^kf = f_0^kE(w_k^2)\circ T^{-k}f$. Then $\langle \left(f_0^{k+1+n}E(w_{k+1+n}^2)\circ T^{-(k+1+n)} - (1+n)\left(f_0^{(k)}E(w_{k+1}^2)E(f_0)\circ T^{-k}\right) + n(f_0^kE(w_k^2)\circ T^{-k})\right)\chi_E, \chi_E\rangle \geq 0$.

Which implies $\int_E \left(f_0^{k+1+n}E(w_{k+1+n}^2)\circ T^{-(k+1+n)}\right) - (1+n)\left(f_0^{(k+1)}E(w_{k+1}^2)E(f_0)\circ T^{-k}\right) + n(f_0^kE(w_k^2)\circ T^{-k})d\lambda \geq 0$ for every E in Σ.

Hence W_T is of k quasi n-class Q^* if and only if $\left(f_0^{k+1+n}E(w_{k+1+n}^2)\circ T^{-(k+1+n)}\right) - (1+n)\left(f_0^{(k+1)}E(w_{k+1}^2)E(f_0)\circ T^{-k}\right) + n(f_0^kE(w_k^2)\circ T^{-k}) \geq 0$ a.e.

Corollary 5.6

Let W_T be a weighted composition operator on $B(L^2(\lambda))$ and assume that $T^{-1}\Sigma = \Sigma$. Then W_T is of k quasi n-class Q^* if and only if $\big(f_0^{k+1+n}(w_{k+1+n}^2) \circ T^{-(k+1+n)}\big) - (1+n)\big(f_0^{(k+1)}(w_{k+1}^2) \circ T^{-k}\big) + n\big(f_0^k(w_k^2) \circ T^{-k}\big) \geq 0$ a.e.

Theorem 5.7
Let W_T be a weighted composition operator on $B(L^2(\lambda))$. Then W_T^* is of k quasi n-class Q^* if and only if $w_{k+1+n}(f_0^{k+1+n} \circ T^{k+1+n})E(w_{k+1+n}) - (1+n)w_k E(w_{k+2})(f_0 E(f_0^k) \circ T^k) + n w_k (f_0^k \circ T^k)E(w_k) \geq 0$ a.e.

Corollary 5.8
Let W_T be a weighted composition operator on $B(L^2(\lambda))$ and $T^{-1}\Sigma = \Sigma$. Then W_T^* is of k quasi n-class Q^* if and only if $w_{k+1+n}^2(f_0^{k+1+n} \circ T^{k+1+n}) - (1+n)w_k w_{k+2}(f_0^{k+1} \circ T^k) + n w_k^2 (f_0^k \circ T^k) \geq 0$ a.e.

The Aluthge transform of T is the operator $\tilde{T}$ given by $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ was introduced in [1] by Aluthge is the. The idea behind the Aluthge transform is to convert an operator into another operator which shares with the first one some spectral properties but it is closed to being a normal operator. More generally we may have family of operators $T_r : 0 < r \leq 1$ where $T_r = |T|^r U|T|^{1-r}$[2. For a composition operator C, the polar decomposition is given by $C = U|C|$ where $|C|f = \sqrt{f_0}f$ and $Uf = \frac{1}{\sqrt{f_0 \circ T}} f \circ T$.
In [11] Lambert has given general Aluthge transformation for composition operator as $C_r = |C|^r U|C|^{1-r}$ and $C_r f = \left(\frac{f_0}{f_0 \circ T}\right)^{\frac{r}{2}} f \circ T$. That is C_r is the weighted composition operators with weights $\pi = \left(\frac{f_0}{f_0 \circ T}\right)^{\frac{r}{2}}$ where $0 < r < 1$. Since C_r is weighted composition operator it is easy to show that $|C_r|f = \sqrt{f_0.(E(\pi)^2 T^{-1})}f$ and $|C_r^*|f = vE[v.f]$ where $\pi = \frac{\pi\sqrt{f_0 \circ T}}{[E(\pi\sqrt{f_0 \circ T})^2]^{\frac{1}{4}}}$. Also we have

$$C_r^k f = \pi_k (f \circ T^k)$$
$$C_r^{*k} = f_0^k E(\pi_k . f) \circ T^{-k}$$
$$C_r^{*k} C_r^k f = f_0^k E(\pi_k^2) \circ T^{-k} f$$

Theorem 5.9
Let $C_r \in B(L^2(\lambda))$. Then C_r is of k quasi n-class Q if and only if $\big(f_0^{k+1+n} E(\pi_{k+1+n}^2) \circ T^{-(k+1+n)}\big) - (1+n)\big(f_0^{(k+1)} E(\pi_{k+1}^2) \circ T^{-(k+1)}\big) + n\big(f_0^k E(\pi_k^2) \circ T^{-k}\big) \geq 0$ a.e.

Proof

Since C_r is a weighted composition operator with weight $= \left(\frac{f_0}{f_0 \circ T}\right)^{\frac{r}{2}}$, it follows from Theorem 5.1, that C_r is of k quasi n-class Q if and only if $\big(f_0^{k+1+n} E(\pi_{k+1+n}^2) \circ T^{-(k+1+n)}\big) - (1+n)\big(f_0^{(k+1)} E(\pi_{k+1}^2) \circ T^{-(k+1)}\big) + n\big(f_0^k E(\pi_k^2) \circ T^{-k}\big) \geq 0$ a.e.

Corollary 5.10
If $T^{-1}\Sigma = \Sigma$ and $C_r \in B(L^2(\lambda))$. Then C_r is of k quasi n-class Q if and only if $\big(f_0^{k+1+n} \pi_{k+1+n}^2 \circ T^{-(k+1+n)}\big) - (1+n)\big(f_0^{(k+1)} \pi_{k+1}^2 \circ T^{-(k+1)}\big) + n\big(f_0^k \pi_k^2 \circ T^{-k}\big) \geq 0$ a.e.

Theorem 5.11
Let $C_r \in B(L^2(\lambda))$. Then C_r^* is of k quasi n-class Q if and only if $\pi_{k+1+n}\big(f_0^{k+1+n} \circ T^{-(k+1+n)}\big)E(\pi_{k+1+n}) - (1+n)\pi_{k+1}\big(f_0^{(k+1)} \circ T^{-(k+1)}\big)E(\pi_{k+1}) + n\pi_k (f_0^k \circ T^{-k})E(\pi_k) \geq 0$ a.e.

Proof

Since C_r^* is a weighted composition operator with weight $= \left(\frac{f_0}{f_0 \circ T}\right)^{\frac{r}{2}}$, it follows from Theorem 5.3, that C_r^* is of k quasi n-class Q if and only if $\pi_{k+1+n}\big(f_0^{k+1+n} \circ T^{-(k+1+n)}\big)E(\pi_{k+1+n}) - (1+n)\pi_{k+1}\big(f_0^{(k+1)} \circ T^{-(k+1)}\big)E(\pi_{k+1}) + n\pi_k (f_0^k \circ T^{-k})E(\pi_k) \geq 0$ a.e.

Corollary 5.12
Let $C_r \in B(L^2(\lambda))$ and $T^{-1}\Sigma = \Sigma$. Then C_r^* is of k quasi n-class Q if and only if $\pi_{k+1+n}^2\big(f_0^{k+1+n} \circ T^{-(k+1+n)}\big) - (1+n)\pi_{k+1}^2\big(f_0^{(k+1)} \circ T^{-(k+1)}\big) + n\pi_k^2 (f_0^k \circ T^{-k}) \geq 0$ a.e.

Theorem 5.13
Let $C_r \in B(L^2(\lambda))$. Then C_r is of k quasi n-class Q^* if and only if $\big(f_0^{k+1+n} E(\pi_{k+1+n}^2) \circ T^{-(k+1+n)}\big) - (1+n)\big(f_0^{(k+1)} E(\pi_{k+1}^2) E(f_0) \circ T^{-(k+1)}\big) + n\big(f_0^k E(\pi_k^2) \circ T^{-k}\big) \geq 0$ a.e.

Proof

Since C_r is a weighted composition operator with weight $= \left(\frac{f_0}{f_0 \circ T}\right)^{\frac{r}{2}}$, it follows from Theorem 7.2, that C_r is of k quasi n-class Q^* if and only if $\left(f_0^{k+1+n}E(\pi_{k+1+n}^2) \circ T^{-(k+1+n)}\right) - (1+n)\left(f_0^{(k+1)}E(\pi_{k+1}^2)E(f_0) \circ T^{-(k+1)}\right) + n\left(f_0^k E(\pi_k^2) \circ T^{-k}\right) \geq 0$ a.e.

Corollary 5.14
If $T^{-1}\Sigma = \Sigma$ and $C_r \in B(L^2(\lambda))$. Then C_r is of k quasi n-class Q^* if and only if $\left(f_0^{k+1+n}\pi_{k+1+n}^2 \circ T^{-(k+1+n)}\right) - (1+n)\left(f_0^{(k+1)}\pi_{k+1}^2 \circ T^{-(k+1)}\right) + n\left(f_0^k \pi_k^2 \circ T^{-k}\right) \geq 0$ a.e.

Theorem 5.15
Let $C_r \in B(L^2(\lambda))$. Then C_r^* is of k quasi n-class Q^* if and only if $\pi_{k+1+n}\left(f_0^{k+1+n} \circ T^{(k+1+n)}\right)E(\pi_{k+1+n}) - (1+n)\pi_{k+1}E(\pi_{k+1})\left(f_0^{(k+1)}E(f_0) \circ T^{(k+1)}\right) + n\pi_k\left(f_0^k \circ T^k\right)E(\pi_k) \geq 0$ a.e.

Corollary 5.16
If $T^{-1}\Sigma = \Sigma$ and $C_r^* \in B(L^2(\lambda))$. Then C_r^* is of k quasi n-class Q^* if and only if $\pi_{k+1+n}^2\left(f_0^{k+1+n} \circ T^{(k+1+n)}\right) - (1+n)\pi_{k+1}^2\left(f_0^{(k+1)} \circ T^{(k+1)}\right) + n\pi_k^2\left(f_0^k \circ T^k\right) \geq 0$ a.e.

B. P Duggal [5] described the second Aluthge Transformation of T by $\tilde{T} = |\hat{T}|^{\frac{1}{2}}V|\hat{T}|^{\frac{1}{2}}$, where $\hat{T} = V|\hat{T}|$ is the polar decomposition of $\hat{T}$. Now we consider $\tilde{C} = |C_r|^{\frac{1}{2}}V|C_r|^{\frac{1}{2}}$, where $C_r = V\,|C_r|$ is the polar decomposition of the generalized Aluthge transformation is C_r: $0 < r < 1$. We have $|C_r|f = \sqrt{J}f$, where $J = f_0.E(\pi^2) \circ T^{-1}$.
$\tilde{C} = |C_r|^{\frac{1}{2}}V|C_r|^{\frac{1}{2}} = \sqrt{J}^{\frac{1}{2}}V\sqrt{J}^{\frac{1}{2}} f = \sqrt{J}^{\frac{1}{2}}\pi\left(\frac{\chi \sup J}{\sqrt{J}} J^{\frac{1}{4}} f\right) \circ T = J^{\frac{1}{4}}\pi\left(\frac{\chi \sup J}{J^{\frac{1}{4}}} \circ T\right)(f \circ T)$. We see then that $\tilde{C}$ is a weighted composition operator with weight $w' = J^{\frac{1}{4}}\pi\left(\frac{\chi \sup J}{J^{\frac{1}{4}}} \circ T\right)$.

Theorem 5.17
If $\tilde{C}$ is of k quasi n-class Q if and only if $\left(f_0^{k+1+n}E(w'^2_{k+1+n}) \circ T^{-(k+1+n)}\right) - (1+n)\left(f_0^{(k+1)}E(w'^2_{k+1}) \circ T^{-(k+1)}\right) + n\left(f_0^k E(w'^2_k) \circ T^{-k}\right) \geq 0$ a.e.

Proof

Since $\tilde{C}$ is a weighted composition operator with weight $w' = J^{\frac{1}{4}}\pi\left(\frac{\chi \sup J}{J^{\frac{1}{4}}} \circ T\right)$, then by Theorem 5.1 we obtain the result.

Corollary 5.18
If $T^{-1}\Sigma = \Sigma$ and $\tilde{C} \in B(L^2(\lambda))$ is of k quasi n-class Q if and only if $\left(f_0^{k+1+n}w'^2_{k+1+n} \circ T^{-(k+1+n)}\right) - (1+n)\left(f_0^{(k+1)}w'^2_{k+1} \circ T^{-(k+1)}\right) + n\left(f_0^k w'^2_k \circ T^{-k}\right) \geq 0$ a.e.

Theorem 5.19
Let $\tilde{C} \in B(L^2(\lambda))$. Then $\tilde{C}^*$ is of k quasi n-class Q if and only if $w'_{k+1+n}\left(f_0^{k+1+n}E(w'^2_{k+1+n}) \circ T^{-(k+1+n)}\right) - (1+n)\,w'_{k+1}\left(f_0^{(k+1)}E\left(w'^2_{k+1}\right) \circ T^{-(k+1)}\right) + n\,w'_k\left(f_0^k E(w'^2_k) \circ T^{-k}\right) \geq 0$ a.e.

Proof

Since $\tilde{C}^*$ is a weighted composition operator with weight $w' = J^{\frac{1}{4}}\pi\left(\frac{\chi \sup J}{J^{\frac{1}{4}}} \circ T\right)$, then by Theorem 5.3 we obtain the result.

Corollary 5.20
Let $\tilde{C} \in B(L^2(\lambda))$ and $T^{-1}\Sigma = \Sigma$. Then $\tilde{C}^*$ is of k quasi n-class Q if and only if $w'^2_{k+1+n}\left(f_0^{k+1+n} \circ T^{-(k+1+n)}\right) - (1+n)\,w'^2_{k+1}\left(f_0^{(k+1)} \circ T^{-(k+1)}\right) + n\,w'^2_k\left(f_0^k \circ T^{-k}\right) \geq 0$ a.e.

Theorem 5.21
If $\tilde{C}$ is of k quasi n-class Q^* if and only if $\left(f_0^{k+1+n}E(w'^2_{k+1+n}) \circ T^{-(k+1+n)}\right) - (1+n)\left(f_0^{(k+1)}E(w'^2_{k+1})E(f_0) \circ T^{-(k+1)}\right) + n\left(f_0^k E(w'^2_k) \circ T^{-k}\right) \geq 0$ a.e.

Corollary 5.22
If $T^{-1}\Sigma = \Sigma$ and $\tilde{C} \in B(L^2(\lambda))$ is of k quasi n-class Q^* if and only if $\left(f_0^{k+1+n}w'^2_{k+1+n} \circ T^{-(k+1+n)}\right) - (1+n)\left(f_0^{(k+1)}w'^2_{k+1} \circ T^{-(k+1)}\right) + n\left(f_0^k w'^2_k \circ T^{-k}\right) \geq 0$ a.e.

Theorem 5.23

Let $\tilde{C} \in B(L^2(\lambda))$. Then $\tilde{C}^*$ is of k quasi n-class Q^* if and only if $w'_{k+1+n}\left(f_0^{k+1+n} \circ T^{(k+1+n)}\right)E(w'_{k+1+n}) - (1+n)w'_{k+1}E(w'_{k+1})\left(f_0^{(k+1)}E(f_0) \circ T^{(k+1)}\right) + nw'_k(f_0^k \circ T^k)E(w'_k) \geq 0$ a.e.

Corollary 5.24
Let $\tilde{C} \in B(L^2(\lambda))$ and $T^{-1}\Sigma = \Sigma$. Then $\tilde{C}^*$ is of k quasi n-class Q^* if and only if $w'^2_{k+1+n}\left(f_0^{k+1+n} \circ T^{(k+1+n)}\right) - (1+n)w'^2_{k+1}\left(f_0^{(k+1)} \circ T^{(k+1)}\right) + nw'^2_k(f_0^k \circ T^k) \geq 0$ a.e.

VI. k QUASI n-CLASS Q OF k QUASI n-CLASS Q^* WEIGHTED COMPOSITION OPERATORS ON WEIGHTED HARDY SPACE.

The set $H^2(\beta)$ of formal complex power series $f(z) = \sum_{m=0}^{\infty} a_m z^m$ such that $\|f\|_\beta^2 = \sum_{m=0}^{\infty} |a_m|^2 \beta_m^2 < \infty$ is a Hilbert space of functions analytic in the unit disc with the inner product.
$\langle f, g\rangle_\beta = \sum_{m=0}^{\infty} a_m \overline{b_m} \beta_m^2$ for an analytic map f on the open unit disc D and $g(z) = \sum_{m=0}^{\infty} b_m z^m$.
Let $\phi: D \to D$ be an analytic self map of the unit disc and consider the corresponding composition operator C_ϕ acting on $H^2(\beta)$. That is $C_\phi(f) = f \circ \phi$ for $f \in H^2(\beta)$. The operators C_ϕ are not necessarily defined on all of $H^2(\beta)$. They are everywhere defined in some special cases on the classical Hardy Space H^2 (the case when $\beta_n = 1$ for all n) and on a general space $H^2(\beta)$ if the function ϕ is analytic on some open set containing the closed unit disc having supremum norm strictly smaller than one. The weighted composition operator W_ϕ is defined as $(W_\phi f)(z) = \pi f(\phi(z))$ and $(W_\phi^* f)(z) = \bar{\pi} f(\phi(z))$ for every $z \in D$.
Let w be a point on the open disc. Define $k_w^\beta(z) = \sum_{m=0}^{\infty} \frac{z^m \overline{w}^{-m}}{\beta_m^2}$. Then the function k_w^β is a point evaluation for $H^2(\beta)$.Then k_w^β is in $H^2(\beta)$ and $\left\|k_w^\beta\right\|^2 = \sum_{m=0}^{\infty} \frac{|w|^{2m}}{\beta_m^2}$. Thus $\|k_w\|$ is an increasing function of$|w|$. If $f(z) = \sum_{m=0}^{\infty} a_m z^m$ then $\langle f, k_w^\beta\rangle = f(w)$ for all f and k_w^β. Hence we can easily seen that $C_\phi^* k_w^\beta = k_{\phi(w)}^\beta$, $W_\phi^* \square_w^\beta = \bar{\pi} k_w^\beta$ and $k_0^\beta = 1$ (the function identically equal to 1).
Now we characterize k quasi n-class Q and k quasi n-class Q^* composition operators on this space as follows.

Theorem 6.1
If C_ϕ is of k quasi n-class Q operator in $H^2(\beta)$, then $C_\phi^{*k+1+n} C_\phi^{k+1+n} - (1+n) C_\phi^{*k+1} C_\phi^{k+1} + n C_\phi^{*k} C_\phi^k \geq 0$.

Proof
For $f \in H^2(\beta)$, consider

$$\begin{aligned}
&\langle (C_\phi^{*k+1+n} C_\phi^{k+1+n} - (1+n) C_\phi^{*k+1} C_\phi^{k+1} + n C_\phi^{*k} C_\phi^k) f, f\rangle \\
&= \langle (C_\phi^{*k+1+n} C_\phi^{k+1+n}) f, f\rangle - (1+n)\langle (C_\phi^{*k+1} C_\phi^{k+1}) f, f\rangle + n\langle (C_\phi^{*k} C_\phi^k) f, f\rangle \\
&= \langle C_\phi^{*k+1+n} f, C_\phi^{k+1+n} f\rangle - (1+n)\langle C_\phi^{*k+1} f, C_\phi^{k+1} f\rangle + n\langle C_\phi^{*k} f, C_\phi^k f\rangle \\
&= \left\|C_\phi^{k+1+n} f\right\|^2 - (1+n)\left\|C_\phi^{k+1} f\right\|^2 + n\left\|C_\phi^k f\right\|^2
\end{aligned}$$

Let $f = k_0^\beta$ then

$$\begin{aligned}
&\langle (C_\phi^{*k+1+n} C_\phi^{k+1+n} - (1+n) C_\phi^{*k+1} C_\phi^{k+1} + n C_\phi^{*k} C_\phi^k) f, f\rangle \\
&= \left\|C_\phi^{k+1+n} k_0^\beta\right\|^2 - (1+n)\left\|C_\phi^{k+1} k_0^\beta\right\|^2 + n\left\|C_\phi^k k_0^\beta\right\|^2 \\
&= \left\|k_0^\beta\right\|^2 - (1+n)\left\|k_0^\beta\right\|^2 + n\left\|k_0^\beta\right\|^2 \\
&= 0
\end{aligned}$$

If C_ϕ is of k quasi n-class Q operator.

Theorem 6.2
If C_ϕ^* is of k quasi n-class Q operator in $H^2(\beta)$, then $C_\phi^{k+1+n} C_\phi^{*k+1+n} - (1+n) C_\phi^{k+1} C_\phi^{*k+1} + n C_\phi^k C_\phi^{*k} \geq 0$.

Proof
For $f \in H^2(\beta)$, consider

$$\begin{aligned}
&\langle (C_\phi^{k+1+n} C_\phi^{*k+1+n} - (1+n) C_\phi^{k+1} C_\phi^{*k+1} + n C_\phi^k C_\phi^{*k}) f, f\rangle \\
&= \left\|C_\phi^{*k+1+n} f\right\|^2 - (1+n)\left\|C_\phi^{*k+1} f\right\|^2 + n\left\|C_\phi^{*k} f\right\|^2
\end{aligned}$$

Let $f = k_0^\beta$ and $\phi(0) = 0$ then we have

$$\begin{aligned}
&\langle (C_\phi^{k+1+n} C_\phi^{*k+1+n} - (1+n) C_\phi^{k+1} C_\phi^{*k+1} + n C_\phi^k C_\phi^{*k}) f, f\rangle \\
&= \left\|C_\phi^{*k+1+n} k_0^\beta\right\|^2 - (1+n)\left\|C_\phi^{*k+1} k_0^\beta\right\|^2 + n\left\|C_\phi^{*k} k_0^\beta\right\|^2 \\
&= \left\|k_0^\beta\right\|^2 - (1+n)\left\|k_0^\beta\right\|^2 + n\left\|k_0^\beta\right\|^2 \\
&= 0
\end{aligned}$$

Hence C_ϕ^* is of k quasi n-class Q operator.

Theorem 6.3
If C_ϕ is of k quasi n-class Q^* operator in $H^2(\beta)$ if and only $\left\|k_0^\beta\right\|^2 \geq \left\|k_{\phi(0)}^\beta\right\|^2$.

Theorem 6.4
If C_ϕ^* is of k quasi n-class Q^* operator in $H^2(\beta)$ if and only $\left\|k_{\phi^{k+1+n}(0)}^{\beta}\right\|^2 \geq \left\|k_{\phi^k(0)}^{\beta}\right\|^2$.

Next we characterize the k quasi n-class Q and k quasi n-class Q^* weighted composition operator on weighted hardy space as follows

Theorem 6.5
An operator $W_\phi \in H^2(\beta)$ is k quasi n-class Q if and only if $\|\pi^{k+1+n}\|^2 - (1+n)\|\pi^{k+1}\|^2 + n\|\pi^k\|^2 \geq 0$.

Proof
Since W_ϕ is k quasi n-class Q operator, then for any $f \in H^2(\beta)$, we have

$$\langle (W_\phi^{*k+1+n} W_\phi^{k+1+n} - (1+n) W_\phi^{*k+1} W_\phi^{k+1} + n W_\phi^{*k} W_\phi^{k}) f, f \rangle \geq 0$$
$$\Leftrightarrow \quad \left\|W_\phi^{k+1+n} f\right\|^2 - (1+n)\left\|W_\phi^{k+1} f\right\|^2 + n\left\|W_\phi^{k} f\right\|^2 \geq 0$$
$$\Leftrightarrow \left\|W_\phi^{k+1+n} k_0^\beta\right\|^2 - (1+n)\left\|W_\phi^{k+1} k_0^\beta\right\|^2 + n\left\|W_\phi^{k} k_0^\beta\right\|^2 \geq 0 \text{ when } f = k_0^\beta$$
$$\Leftrightarrow \quad \left\|\pi^{k+1+n} k_0^\beta\right\|^2 - (1+n)\left\|\pi^{k+1} k_0^\beta\right\|^2 + n\left\|\pi^{k} k_0^\beta\right\|^2 \geq 0$$
$$\Leftrightarrow \|\pi^{k+1+n}\|^2 \left\|k_0^\beta\right\|^2 - (1+n)\|\pi^{k+1}\|^2 \left\|k_0^\beta\right\|^2 + n\|\pi^k\|^2 \left\|k_0^\beta\right\|^2 \geq 0$$
$$\Leftrightarrow \|\pi^{k+1+n}\|^2 - (1+n)\|\pi^{k+1}\|^2 + n\|\pi^k\|^2 \geq 0.$$

Theorem 6.6
An operator $W_\phi^* \in H^2(\beta)$ is k quasi n-class Q if and only if $\|\bar{\pi}^{k+1+n}\|^2 - (1+n)\|\bar{\pi}^{k+1}\|^2 + n\|\bar{\pi}^k\|^2 \geq 0$.

Proof
Since W_ϕ^* is k quasi n-class Q operator, we have

$$\langle (W_\phi^{k+1+n} W_\phi^{*k+1+n} - (1+n) W_\phi^{k+1} W_\phi^{*k+1} + n W_\phi^{k} W_\phi^{*k}) f, f \rangle \geq 0 \text{ for any } f \in H^2(\beta)$$
$$\langle (W_\phi^{k+1+n} W_\phi^{*k+1+n} - (1+n) W_\phi^{k+1} W_\phi^{*k+1} + n W_\phi^{k} W_\phi^{*k}) f, f \rangle \geq 0$$
$$\Leftrightarrow \left\|W_\phi^{*k+1+n} f\right\|^2 - (1+n)\left\|W_\phi^{*k+1} f\right\|^2 + n\left\|W_\phi^{*k} f\right\|^2 \geq 0$$
$$\Leftrightarrow \left\|\bar{\pi}^{k+1+n} k_0^\beta\right\|^2 - (1+n)\left\|\bar{\pi}^{k+1} k_0^\beta\right\|^2 + n\left\|\bar{\pi}^{k} k_0^\beta\right\|^2 \geq 0 \text{ for } f = k_0^\beta \text{ and } \phi(0) = 0$$
$$\Leftrightarrow \|\bar{\pi}^{k+1+n}\|^2 - (1+n)\left\|\square^{k+1}\right\|^2 + n\|\bar{\pi}^k\|^2 \geq 0.$$

Hence the theorem.

Theorem 6.7
An operator W_ϕ is of k quasi n-class Q^* operator in $H^2(\beta)$ if and only if $\|\pi^{k+1+n}\|^2 - (1+n)|\pi|^2\|\pi^{k-1}\|^2 + n\|\pi^k\|^2 \geq 0$.

Theorem 6.8
An operator $W_\phi^* \in H^2(\beta)$ is of k quasi n-class Q^* if and only if $\|\bar{\pi}^{k+1+n}\|^2 - (1+n)|\pi|^2\|\bar{\pi}^{k-1}\|^2 + n\|\bar{\pi}^k\|^2 \geq 0$.

VII. (n,k) QUASI CLASS Q AND (n,k) QUASI CLASS Q^* OPERATORS

As composite multiplication operator to a linear transformation acting on a set of complex value Σ measurable functions f of the form $M_{u,T}(f) = C_T M_u f = u \circ T f \circ T$ where u is a complex valued Σ measurable function. In the case $u = 1$a.e, $M_{u,T}$ becomes a composition operator denoted by C_T.

Proposition 7.1
Let the composite multiplication operator $M_{u,T}(f) \in B(L^2(\lambda))$ then for $u \geq 0$
(1) $M_{u,T}^* M_{u,T} f = u^2 f_0 f$.
(2) $M_{u,T} M_{u,T}^* f = (u^2 \circ T)(f_0 \circ T).E(f)$.

Since $M_{u,T}(f) = C_T M_u f = u \circ T f \circ T$, $M_{u,T}^n(f) = (C_T M_u)^n f = u^n (f \circ T)^2$, $M_{u,T}^*(f) = u f_0 . E(f) \circ T^{-1}$ and $M_{u,T}^{*n}(f) = u f_0 . E(u f_0) \circ T^{-(n-1)} . E(f) \circ T^{-n}$ where $E(u f_0) \circ T^{-(n-1)} = E(u f_0) \circ T^{-1}, E(u f_0) \circ T^{-2}, \ldots, E(u f_0) \circ T^{-(n-1)}$, $E(u f_0) \circ T^{n-1} = E(u f_0) \circ T^1, E(u f_0) \circ T^2, \ldots, E(u f_0) \circ T^{n-1}$.

In this section, we study k quasi n-class Q and k quasi n-class Q^* composite multiplication operator as follows.

Theorem 7.2
Let the composite multiplication operator $M_{u,T}(f) \in B(L^2(\lambda))$. Then $M_{u,T}$ is k quasi n-class Q if and only if $u f_0 . E(u f_0) \circ T^{-(k+n)} E(u_{k+1+n}) \circ T^{-(k+1+n)} - (1+n) u f_0 . E(u f_0) \circ T^{-k} E(u_{k+1}) \circ T^{-(k+1)} + n u f_0 . E(u f_0) \circ T^{-(k-1)} E(u_k) \circ T^{-k} \geq 0$. a.e.

Proof

Suppose $M_{u,T}$ is k quasi n-class Q operator, then

$M_{u,T}^{*k+1+n}M_{u,T}^{k+1+n} - (1+n)M_{u,T}^{*k+1}M_{u,T}^{k+1} + nM_{u,T}^{*k}M_{u,T}^{k} \geq 0$. Then for any $f \in L^2(\lambda)$, we have

$$\langle\left(M_{u,T}^{*k+1+n}M_{u,T}^{k+1+n} - (1+n)M_{u,T}^{*k+1}M_{u,T}^{k+1} + nM_{u,T}^{*k}M_{u,T}^{k}\right)f, f\rangle \geq 0$$

$$\langle M_{u,T}^{*k+1+n}M_{u,T}^{k+1+n}f, f\rangle - (1+n)\langle M_{u,T}^{*k+1}M_{u,T}^{k+1}f, f\rangle + n\langle M_{u,T}^{*k}M_{u,T}^{k}f, f\rangle \geq 0$$

Since $M_{u,T}^{*k}M_{u,T}^{k} = uf_0.E(uf_0)\circ T^{-(k-1)}.E(f)\circ T^{-n}$, $M_{u,T}^{k}M_{u,T}^{*k} = u_k u\circ T^k.f_0\circ T^k.E(uf_0)\circ T^{k-1}.E(f)$. where $u_k = u\circ T.u\circ T^2\ldots u\circ T^k$

$$\Leftrightarrow uf_0.E(uf_0)\circ T^{-(k+n)}E(u_{k+1+n})\circ T^{-(k+1+n)} - (1+n)uf_0.E(uf_0)\circ T^{-k}$$
$$E(u_{k+1})\circ T^{-(k+1)} + nuf_0.E(uf_0)\circ T^{-(k-1)}E(u_k)\circ T^{-k} \geq 0$$

Corollary 7.3

If the composition operator $C_T \in B\left(L^2(\lambda)\right)$ then C_T is k quasi n-class Q if and only if $f_0.E(f_0)\circ T^{-(k+n)} - (1+n)f_0.E(f_0)\circ T^{-k} + nf_0.E(f_0)\circ T^{-(k-1)} \geq 0$. a.e.

Proof

By putting $u = 1$ in Theorem 7.2, we get the result.

Theorem 7.4

Let the composite multiplication operator $M_{u,T}(f) \in B\left(L^2(\lambda)\right)$. Then $M_{u,T}^*$ is k quasi n-class Q if and only if $u_{k+1+n}u\circ T^{k+1+n}f_0\circ T^{k+1+n}.E(uf_0)\circ T^{(k+n)} - (1+n)u_{k+1}(u\circ T^{k+1})(f_0\circ T^{k+1}).E(uf_0)\circ T^{(k)} + n(u_k u\circ T^k)(f_0\circ T^k).E(f_0)\circ T^{k-1} \geq 0$. a.e.

Corollary 7.5

If the composition operator $C_T \in B\left(L^2(\lambda)\right)$ then C_T^* is k quasi n-class Q if and only if $f_0\circ T^{k+1+n}.E(f_0)\circ T^{k+n} - (1+n)f_0\circ T^{k+1}.E(f_0)\circ T^k + nf_0\circ T^k.E(f_0)\circ T^{k-1} \geq 0$. a.e.

Theorem 7.6

Let the composite multiplication operator $M_{u,T}(f) \in B\left(L^2(\lambda)\right)$. Then $M_{u,T}$ is k quasi n-class Q^* if and only if $uf_0.E(uf_0)\circ T^{-(k+n)}E(u_{k+1+n})\circ T^{-(k+1+n)} - (1+n)uf_0.E(uf_0)\circ T^{-(k-1)}E(u_{k+2})\circ T^{-k} + nuf_0.E(uf_0)\circ T^{-(k-1)}E(u_k)\circ T^{-k} \geq 0$. a.e.

Corollary 7.7

If the composition operator $C_T \in B\left(L^2(\lambda)\right)$ then C_T is k quasi n-class Q^* if and only if $f_0.E(f_0)\circ T^{-(k+n)} - (1+n)f_0E(f_0)\circ T^{-(k-1)}.E(f_0)\circ T^{-k} + nf_0.E(f_0)\circ T^{-(k-1)} \geq 0$. a.e.

Theorem 7.8

Let the composite multiplication operator $M_{u,T}(f) \in B\left(L^2(\lambda)\right)$. Then $M_{u,T}^*$ is k quasi n-class Q^* if and only if $u_{k+1+n}uf_0\circ T^{k+1+n}.E(uf_0)\circ T^{-(k+n)} - (1+n)u_k uf_0.E(u^3f_0^2)\circ T + n(u_k uf_0\circ T^k).E(uf_0)\circ T^{-(k-1)} \geq 0$. a.e.

Corollary 7.9

If the composition operator $C_T \in B\left(L^2(\lambda)\right)$ then C_T^* is k quasi n-class Q^* if and only if $f_0\circ T^{k+1+n}.E(f_0)\circ T^{-(k+n)} - (1+n)f_0.E(f_0^2)\circ T + nf_0\circ T^k.E(f_0)\circ T^{-(k-1)} \geq 0$. a.e.

VIII. ALUTHGE TRANSFORMATION OF k QUASI n-CLASS Q OF k QUASI n-CLASS Q^* OPERATOR

Let $T = U|T|$ be the polar decomposition of T. Then the Aluthge transformation $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ was introduced by Aluthge[1]. An operator T is called w hyponormal if $|\tilde{T}| \geq |T| \geq |\tilde{T}^*|$ and he defined $\tilde{\tilde{T}} = |\tilde{T}|^{\frac{1}{2}}\tilde{U}|\tilde{T}|^{\frac{1}{2}}$ where $\tilde{T} = \tilde{U}|\tilde{T}|$. Also the adjoint of aluthge transformation is defined $\tilde{T}^* = |T|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}}$, $*$-Aluthge transformation is $\tilde{T}^* = |T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}}$, and adjoint of $*$-Aluthge transformation is given by $\tilde{T}^{*^*} = |T^*|^{\frac{1}{2}}U^*|T^*|^{\frac{1}{2}}$.

Theorem 8.1

An operator T is k quasi n class Q if and only if $(1+n)T^{*k}|T|^2T^k \leq T^{*k}\left|T^{(1+n)}\right|^2T^k + nT^{*k}T^k$ for all $x \in H$ and for every positive integer n.

Proof

Since T is k quasi n class Q operator, then $T^{*k}\left(T^{*(1+n)}T^{(1+n)} - (1+n)T^*T + nI\right)T^k \geq 0$ for every positive integer n. By simple calculation we get the result.

Theorem 8.2

If $T = U|T|$is the polar decomposition of k quasi n class Q operator T, then T is k quasi n class Q operator.

Theorem 8.3
If T is k quasi n class Q operator T and S is unitary such that $TS = ST$ then $A = TS$ is also k quasi n class Q operator.

Theorem 8.4
Let $T = U|T|$ be the polar decomposition of k quasi n class Q operator T, where U is unitary if and only if $\tilde{T}$ is k quasi n class Q operator.

Proof
Suppose we assume that T is k quasi n class Q operator and $T = U|T|$ is the polar decomposition of T, then we have that $T^{*k}\left(T^{*(1+n)}T^{(1+n)} - (1+n)T^*T + nI\right)T^k \geq 0$ for every positive integer n.

$$\Leftrightarrow (U|T|)^{*k}\left((U|T|)^{*(1+n)}(U|T|)^{(1+n)} - (1+n)(U|T|)^*(U|T|) + nI\right)(U|T|)^k \geq 0$$

$$\Leftrightarrow |T^k|^{\frac{1}{2}}U^*|T^k|^{\frac{1}{2}}(|T^{(1+n)}|^{\frac{1}{2}}U^{*(1+n)}|T^{*(1+n)}|U^{(1+n)}|T^{(1+n)}|^{\frac{1}{2}} - (1+n)$$
$$|T|^{\frac{1}{2}}U^*|T^*|U|T|^{\frac{1}{2}} + nI)|T^k|^{\frac{1}{2}}U^k|T^k|^{\frac{1}{2}} \geq 0$$
$$\Leftrightarrow \quad \tilde{T}^{*k}\left(\tilde{T}^{*(1+n)}\tilde{T}^{(1+n)} - (1+n)\tilde{T}^*\tilde{T} + nI\right)\tilde{T}^k \geq 0$$

for every positive integer n. Hence $\tilde{T}$ is k quasi n class Q operator.

Theorem 8.5
Let $T = U|T|$ be the polar decomposition of k quasi n class Q operator $T$and U is unitary, then T is k quasi n class Q if and only if $\tilde{T}^*$ is k quasi n class Q operator.

Proof
Suppose we assume that T is k quasi n class Q operator and $T = U|T|$ is the polar decomposition of T, then we have that $T^{*k}\left(T^{*(1+n)}T^{(1+n)} - (1+n)T^*T + nI\right)T^k \geq 0$ for every positive integer n.

$$\Leftrightarrow (U|T|)^{*k}\left((U|T|)^{*(1+n)}(U|T|)^{(1+n)} - (1+n)(U|T|)^*(U|T|) + nI\right)(U|T|)^k \geq 0$$

$$\Leftrightarrow |T^k|^{\frac{1}{2}}U^*|T^k|^{\frac{1}{2}}(|T^{(1+n)}|^{\frac{1}{2}}U^{*(1+n)}|T^{*(1+n)}|U^{(1+n)}|T^{(1+n)}|^{\frac{1}{2}} - (1+n)$$
$$|T|^{\frac{1}{2}}U^*|T^*|U|T|^{\frac{1}{2}} + nI)|T^k|^{\frac{1}{2}}U^k|T^k|^{\frac{1}{2}} \geq 0$$
$$\Leftrightarrow \quad \tilde{T}^{*k}\left(\tilde{T}^{(1+n)}\tilde{T}^{*(1+n)} - (1+n)\tilde{T}\,\tilde{T}^* + nI\right)\tilde{T}^k \geq 0$$

for every positive integer n. Hence $\tilde{T}^*$ is k quasi n class Q operator.

Corollary 8.6
If that $\tilde{T}$ is k quasi n class Q if and only if that $\tilde{T}^*$ is k quasi n class Q operator.

Theorem 8.7
Let $T = U|T|$ be the polar decomposition of k quasi n class Q operator T and U is unitary, then that T is k quasi n class Q if and only if that $\tilde{T}^{**}$ is k quasi n class Q operator.

Theorem 8.8
Let $T = U|T|$ be the polar decomposition of k quasi n class Q operator T and U is unitary, then that $\tilde{T}^*$ is k quasi n class Q if and only if that $\tilde{T}^{**}$ is k quasi n class Q operator.

Theorem 8.9
An operator that T is k quasi n class Q^* if and only if $(1+n)T^{*k}|T^*|^2T^k \leq T^{*k}\left|T^{(1+n)}\right|^2T^k + nT^{*k}T^k$ for all $x \in H$ and for every positive integer n.

Theorem 8.10
If $T = U|T|$ is the polar decomposition of k quasi n class Q^* operator T, then that T is k quasi n class Q^* operator.

Theorem 8.11
If that T is k quasi n class Q^* operator T and S is unitary such that $TS = ST$ then $A = TS$ is also k quasi n class Q^* operator.

Theorem 8.12
If that $\tilde{T}$ is k quasi n class Q^* if and only if that $\tilde{T}^*$ is k quasi n class Q^* operator.

Theorem 8.13
If that $\tilde{T}^*$ is k quasi n class Q^* if and only if that $\tilde{T}^{**}$ is k quasi n class Q^* operator.

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