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Nonlinear first order differential equations with anti-periodic boundary value problem

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Abstract - In this paper we illustrate the new existence results for a nonlinear amti-periodic first order problem using a Leray-Schauder alternative. We give the definitions of upper and lower solution. Coupled upper and lower solution are presented. We show the validity of the upper and lower solution method. And also we generate a sequence of approximate solutions converging to a solution of the anti-periodic problem.

Keywords - Anti-periodic boundary value problem, Leray-Schauder alternative, upper and lower solutions, Greens function.

I.INTRODUTION

In this paper we study an anti-periodic for first order differential equation. Anti-periodic problem have been studied extensively in the last ten year.

Sometimes we have a connection between anti-periodic and periodic problem. For example any T-antiperiodic solution gives rise to a 2T-periodic solution if the nonlinearity f satisfies some symmetry condition.

Considre the following nonlinear anti-periodic boundary value problem

$$z'(t) = f(t,z(t)) \text{, a.e. } t \in I ,$$
$$z(0) = -z(T), \quad T > 0 \text{ and } I = [O,T] \qquad (1)$$

Where $f: I \times \mathbb{R} \to \mathbb{R}$ is a L' −Caratheodory function, i.e. , f satisfies

- For every $x \in \mathbb{R}, f(\cdot, x)$ is Lebesgue measurable on$I$.
- For a.e. $t \in I, f(t, \cdot)$ is continuous on$\mathbb{R}$.
- For every $R > 0$ there exists $\varphi \in L'(I)$ such that $|f(t,x)| \leq \varphi(t)$ for a.e. $t \in I$ and all $x \in \mathbb{R}$ with $|x| \leq R$.

Throughout this paper, $C(I)$ denotes the space of continuous functions on I and $AC(I)$ the of absolutely continuous functions on I. For $z \in C(I)$ we consider the usual norm

$$\| z \|_0 = sup_{t\in I}|z(t)|.$$

In the space $C(I)$ we also consider the usual pointwise partial ordering. In such a case we define the interval

$$[p,q] = \{z \in C(I): p \leq z \leq q\}.$$

We say that a functions $z: I \to \mathbb{R}$ is a solution to (1) if $z \in AC(I)$ and it solves (1).

II. BASIC EXISTENCE THEORY

Let $\delta \in \mathbb{R}, F: I \times \mathbb{R} \to \mathbb{R}$ a L' − Caratheodory function and consider the problem

$$z'(t) + \delta z(t) = F\big(t, z(t)\big), \text{ a.e. } t \in I,$$
$$z(0) = -z(T). \qquad (2)$$

If $F(t,z) = f(t,z) + \delta z$ and z is a solution to (2) then z is a solution to (1). Furthermore, it is easy to show that solving (2) is equivalent to finding a $z \in C(I)$ that satisfies $z = Az$. Here $A: C(I) \to C(I)$ is given by

$$[Az](t) = \int_0^T g(t,s)F\big(s, z(s)\big)ds, \qquad (3)$$

Where g is the Green's function

$$g(t,s) = \begin{cases} \dfrac{e^{\delta(T-t+s)}}{e^{\delta T}+1}, & 0 \leq s \leq t \leq T \\ \dfrac{-e^{\delta(s-t)}}{e^{\delta T}+1}, & 0 \leq t < s \leq T \end{cases}$$

Note:

If $F(t,u) = \sigma(t)$ problem (2) is linear and solvable for each $\delta \in \mathbb{R}$ and the solution is given by expression (3).

Theorem 1

Let C be a complete convex subset of a locally convex Hausdorff linear topological space E and U an open subset of C with $p \in U$. In addition let $F: \overline{U} \to C$ be a continuous, compact map. Then either

(A1) F has a fixed point in $\overline{U}$; or

(A2) there is a $z \in \partial U$ and $\mu \in (0,1)$, with $z = \mu F(z) + (1 - \mu)p$.

In addition to applying the Leray-Schauder alternative, we will need a compactness criterion for a set $S \subset PC\,(J, R^n)$.

Theorem 2

Suppose that there exist a continuous and nondecreasing function $\psi: [0,\infty) \to (0,\infty)$ and a function $Q \in L^1(I)$ with

$$|F(t,z)| \leq Q(t)\psi(|z|), \text{ for a.e.} t \in I \text{ and all } z \in \mathbb{R}.$$

In addition suppose that

$$sup_{c\geq 0} \frac{c}{\psi(c)} > k_0 \tag{4}$$

With

$$k_0 = sup_{t\in I} \int_0^T |g(t,s)|Q(s)ds.$$

Then (1) has at least one solution in $AC(I)$.

Proof:

Form (4) there exists $M > 0$ with

$$\frac{M}{\psi(M)} > k_0. \tag{5}$$

For $\mu \in (0,1)$, let $u \in AC(I)$ be any solution of (4). Then, for $t \in I$ we have

$$z(t) = \mu \int_0^T g(t,s)F\big(s,z(s)\big)ds$$

And so

$$|z(t)| \leq \mu \int_o^T \big|g(t,s)F\big(s,z(s)\big)\big|ds$$
$$\leq \int_0^T |g(t,s)|Q(s)\psi(|z(s)|)ds$$
$$\leq (\| z \|_0) \int_0^T |g(t,s)|Q(s)ds.$$

Consequently,

$$\| z \|_0 \leq\ k_0\psi(\| z \|_0)$$

And so

$$\| z \|_0 \neq M \text{ form (5).}$$

III. UPPER AND LOWER SOLUTIONS

The following definition are lower and upper solution is presented.

Definition: 1

We say that a pair of functions

$$\alpha, \beta \in AC(I)$$

are related lower and upper solutions for the anti-periodic problem (1) if

$$\alpha(t) \leq \beta(t),\ \ t \in I \tag{6}$$

$$\alpha'(t) \leq f\big(t,\alpha(t)\big), \text{ a.e. } t \in I\,,\ \ \alpha(0) \leq -\beta(T), \tag{7}$$

and

$$\beta'(t) \geq f\big(t,\beta(t)\big), \text{ a.e. } t \in I\,,\ \beta(0) \geq -\alpha(T). \tag{8}$$

The following is the first result, to our knowledge, that establishes the validity of the lower and upper solutions method for (1) without monotone criteria.

Theorem: 3

Suppose that there exist $\alpha, \beta \in AC(I)$ related lower and upper solutions for (1). Then (1) has at least one solution between $\alpha \text{ and } \beta$.

Proof:

Let $\lambda > 0$ and consider the modified problem

$$z'(t) + \delta z(t) = F^*\big(t,z(t)\big), \text{ a.e. } t \in I,$$
$$z(0) = -z(T), \tag{9}$$

With

$$F^*(t,z) = \begin{cases} f\big(t,\beta(t)\big) +\ \delta\beta, & if\ \beta(t) < z \\ f(t,z) + \delta z, & if\ \alpha(t) \leq z \leq \beta(t) \\ f\big(t,\alpha(t)\big) + \delta\alpha(t), & if\ z < \alpha(t). \end{cases}$$

Then, by the Schauder fixed point theorem, we conclude that (9) has a solution u, since in this case the operator A defined in (3) is continuous and compact.

Now we will show that this solution u satisfies $\alpha(t) \leq z(t) \leq \beta(t)$ for $t \in I$. Assume that $z - \beta$ attains a positive maximum on I at s_0. We shall consider two cases:

Case 1. $s_0 \in (0,T]$.

Then there exist $ק \in (0,s_0)$ such that

$$0 \leq z(t) - \beta(t) \leq z(s_0) - s_0,\ \text{ for all } t \in [ק, s_0].$$

This yields a contradiction, since

$$\beta(s_0) - \beta(ק) \leq z(s_0) - z(ק)$$
$$= \int_ק^{s_0} [f(s,\beta(s)) - \delta(z(s) - \beta(s))]ds$$

$$< \int_{\daleth}^{s_0} \beta'(s)ds$$
$$= \beta(s_0) - \beta(\daleth).$$

Case 2. $s_0 = 0$.

Then $0 < z(0) - \beta(0)$. Note also that $z(T) - \alpha(T) < 0$. Since

$$z(T) = -Z(0) < -\beta(0) \leq \alpha(T).$$

Moreover by hypothesis $\alpha(0) \leq \beta(0) < z(0)$. Therefore, there exist $\daleth \in (0, T)$ with $z(t) - \alpha(t) < 0$ for all $t \in (\daleth, T]$ and $z(\daleth) - \alpha(\daleth) = 0$. Now we have

$$z(T) - z(\daleth) = \int_{\daleth}^{T} [f(s, \alpha(s)) + \delta(\alpha(s) - z(s))]ds$$
$$> \int_{\daleth}^{T} \alpha'(s)ds$$
$$= \alpha(T) - \alpha(\daleth)$$

Which contradicts

$$z(T) - \alpha(T) < 0.$$

Consequently,

$z(t) \leq \beta(t)$ for all$t \in I$. Similarly, we can show that $\alpha \leq z$ on I.

IV. COUPLED UPPER AND LOWER SOLUTIONS

We recall the equation (2) with $F(t,z) = f(t,z) + \delta z$ and we consider again the operator A defined in (3). Note that g is not of constant sign on $I \times I$. Hence. $g = g^- - g^-$ with

$$g^+(t,s) = \max\{g(t,s), 0\}$$

and

$$g^-(t,s) = \max\{-g(t,s), 0\}$$

And we can write the operator given in (3) as

$$[Az](t) = \int_0^T g^+(t,s)F(s, z(s))ds - \int_0^T g^-(t,s)F(s, z(s))ds, \qquad (10)$$

or equivalently as

$$[Az](t) = \int_0^t \frac{e^{\delta(T-t+s)}}{e^{\delta T}+1} F(s, z(s))ds - \int_t^T \frac{e^{\delta(s-t)}}{e^{\delta T}+1} F(s, z(s))ds.$$

Motivated by the expression (10) and the results of [7] we introduce the following operators. For $\tau \in C(I), t \in I$, we define

$$[A^+\tau](t) = \int_0^T g^+(t,s)F(s, \tau(s))ds,$$

and

$$[A^-\tau](t) = \int_0^T g^-(t,s)F(s, \tau(s))ds.$$

Note:

$A^+: C(I) \to C(I)$ and $A^-: C(I) \to C(I)$ are continuous and compietely continuous.

Definition:2

We say that a pair of functions $\alpha, \beta \in C'(I)$ and coupled lower and upper solutions for the anti-periodic problem (1) if (6) holds and

$$\alpha \leq A^+\alpha - A^-\beta, \qquad (11)$$

and

$$\beta \geq A^+\beta - A^-\alpha. \qquad (12)$$

The relation between both definitions is given by the following result.

Theorem: 4

Suppose that α, β are a pair of related lower and upper solutions for the anti-periodic problem (1). Then α, β are a pair of coupled lower and upper solutions for (1). In other words, if α, β are lower and upper solutions in the sense of Definition 1, then they are lower and upper solutions in the sense of Definition 2.

Proof : For every $t \in T$, we have that

$$[A^+\alpha](t) - [A^-\beta](t) = \int_0^T g^+(t,s)F(s, \alpha(s))ds - \int_0^T g^-(t,s)F(s, \beta(s))ds$$

By using the theorem 4 ,equation (11) and the definition of coupled lower and upper solutions, we get

$$\frac{e^{\delta T}}{e^{\delta T}+1}\alpha(t) + \frac{1}{e^{\delta T}+1}\alpha(t) = \alpha(t).$$

Therefore, (12) holds. The validity of (13) is proved analogously.

V. CONCLUSIONS

In this paper, we introduced new comparison theorems for investigation of the anti-periodic boundary value problem for nonlinear first order ordinary differential equations. Our techniques permits to relax restrictions usually imposed on the coupled upper and lower solution. So that our results are of high generality.

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