Solving volterra integral equations by using differential transform method

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Abstract - In this paper, Differential Transform Method (DTM) has been used to solve the Volterra integral equations. We calculated the approximate solution in the form of a series with easily computable terms. Using theis method, we find the exact solutions of linear and nonlinear volterra integral equations. Numerical examples are given. To illustrate the reliability and the performance of the differential transform method.

Keywords - volterra integral equation, differential transform method.

I.INTRODUCTION

We consider the Volterra integral equation of the first kind is an integral equation of the form

$$f(y) = \int_a^y K(y,t)u(t)dt, \qquad (1)$$

and the Volterra integral equation of the second kind is an integral equation of the form

$$u(y) = f(y) + \int_a^y K(y,t)u(t)dt. \qquad (2)$$

where $u(y)$ be the function to be solved for, $f(y)$ is a given known function, and $K(y,t)$ is a known integral kernel.

In this paper, we apply the DTM to solve the linear and nonlinear Volterra integral equation in addition with separable kernels, i.e.

$$K(y,t) = \sum_{i=0}^{N} M_i(y)N_i(t).$$

In this case the nonlinear Volterra integral equation of the first kind or the nonlinear Volterra integral equation of the second kind can be written in the following general form:

$$u(y) = f(y) + \sum_{i=0}^{N} k_i(y) \int_a^y v_i\big(t, u(t)\big)dt. \qquad (3)$$

Using the DTM, several examples of linear and nonlinear Volterra integral equations are tested, and the results reveal that the DTM is very effective and simple.

II.DIFFERENTIAL TRANSFORM METHOD

The differential transform method is a numerical method for solving differential equations. The concept of differential transform method was first introduced by Zhou[3] in 1987, who solved linear and nonlinear initial value problems in electric circuit analysis. This method gives exact values of the nth derivative of an analytical function at a point in terms of known and unknown boundary conditions in a fast manner. The main advantage of differential transformation from Laplace and Fourier transformations is that it can be applied easily to linear equations with constant and variable coefficients and some nonlinear equations.

The differential transformation of the kth derivative of function $f(y)$ is defined as follows:

$$F(k) = \frac{1}{k}\left[\frac{d^k f(y)}{dy^k}\right]_{y=y_0}, \qquad (4)$$

where $f(y)$ is the original function and $F(k)$ is the transformed function. The differential inverse transform of $F(k)$ is defined as

$$f(y) = \sum_{k=0}^{\infty} F(k)(y-y_0)^k. \qquad (5)$$

From equations (4) and (5), we get

$$f(y) = \sum_{k=0}^{\infty} \frac{(y-y_0)^k}{k!}\left[\frac{d^k f(y)}{dy^k}\right]_{y=y_0}, \qquad (6)$$

which implies that the concept of differential transform is derived from Taylor series expansion, but the method does not evaluate the derivatives symbolically. However, relative derivatives are calculated by an iterative way which are described by the transformed equations of the original function. In real applications, the function $f(x)$ is expressed by a finite series and equation (5) can be written as

$$f(y) = \sum_{k=0}^{n} F(k)(y - y_0)^k, \qquad (7)$$

Here n is decided by the convergence of natural frequency. The following table is the fundamental operations performed by differential transform can readily be obtained and are listed below.

Table 1

Operations of differential transformation

Original function	Transformed function
$f(y) = u(y) \pm v(y)$	$F(k) = U(k) \pm v(k)$
$f(y) = \alpha u(y)$	$F(k) = \alpha U(k)$
$f(y) = u(y)v(y)$	$F(k) = \sum_{l=0}^{k} U(l)V(k-l)$
$f(y) = \frac{du(y)}{dx}$	$F(k) = (k+1)U(k+1)$
$f(y) = \frac{d^m u(y)}{dy^m}$	$F(k) = (k+1) \ldots (k+m)U(k+m)$
$f(y) = \int_{x_0}^{x} u(t)dt$	$F(k) = \frac{U(k-1)}{k}, k \geq 1, F(0) = 0$
$f(y) = x^m$	$F(k) = \delta(k-m)$
$f(y) = \exp(\lambda y)$	$F(k) = \frac{\lambda^k}{k!}$
$f(y) = \sin(wy + \alpha)$	$F(k) = \frac{w^k}{k!}\sin(\pi k/2 + \alpha)$
$f(y) = \cos(wx + \alpha)$	$F(k) = \frac{w^k}{k!}\cos(\pi k/2 + \alpha)$

In the following theorem, we find the transformation for two types of product of single-valued functions. These results are very useful on our approach for solving integral equations.

Theorem 1. Suppose that $U(k)$, $V(k)$ and $G(k)$ are the differential transformations of the functions $u(y)$, $v(y)$ and $g(y)$ respectively, then we have the following properties:

(a) If $f(y) = \int_{y_0}^{y} v(t)u(t)dt$, then

$$F(k) = \sum_{l=0}^{k-1} V(l)\frac{U(k-l-1)}{k}, \; F(0) = 0. \qquad (8)$$

(b) If $f(y) = g(y)\int_{y_0}^{y} u(t)dt$, then

$$F(k) = \sum_{l=0}^{k-1} G(l)\frac{U(k-l-1)}{k-l}, \quad F(0) = 0. \qquad (9)$$

Proof.

The proof follows immediately from the equations (1) and (2) and the operations of the differential transformation given in the table 1.

III. NUMERICAL RESULTS

In order to illustrate the advantages and the accuracy of the DTM for solving the linear and nonlinear Volterra integral equation with separable kernels, we have applied the method to differential integral equations.

Example 1. Consider the linear Volterra integral equation

$$u(y) = y + \int_0^x (t-y)u(t)dt, \quad 0 < x < 1. \qquad (10)$$

According to theorem 1 and to the operations of differential transformation givenin table 1, we have the following recurrence relation:

$$U(k) = \delta(k-1) + \sum_{l=0}^{k-l} \delta(l-l)\frac{U(k-l-1)}{k} - \sum_{l=0}^{k-1} \delta(l-1)\frac{U(k-l-1)}{k-l},$$
$$k \geq 1, \qquad U(0) = 0 \qquad (11)$$

Consequesently, we find

$$U(1) = 1, U(2) = 0,$$
$$U(3) = -\frac{1}{3!}, U(4) = 0,$$
$$U(5) = \frac{1}{5!}, U(6) = 0,$$
$$U(7) = -\frac{1}{7!}, U(8) = 0, \ldots$$

Therefore from equation (5), the solution of the integral equation (10) is given by

$$U(y) = y - \frac{1}{3!}y^3 + \frac{1}{5!}y^5 - \frac{1}{7!}y^7 + \cdots = \sin(y),$$

Which is the exact solution of the integral equation(10).

Example 2. Consider the linear Volterra integral equation

$$u(y) = 1 - y - \frac{y^2}{2} + \int_0^y (y-t)u(t)dt, \;\; 0 < y < 1. \qquad (12)$$

According to theorem 1 and to the operations of differential transformation given in table 1, we have the following recurrence relation :

$$U(k) = \delta(k) - \delta(k-1) - \frac{\delta(k-2)}{2} + \sum_{l=0}^{k-1} \delta(l-1)\frac{U(k-l-1)}{k-l} - \sum_{l=0}^{k-1} \delta((l-1)\frac{U(k-l-1)}{k},$$
$$k \geq 1, \qquad U(0) = 1. \qquad (13)$$

Consequently, we find

$$U(1) = -1, U(2) = 0,$$
$$U(3) = -\frac{1}{3!}, U(4) = 0,$$
$$U(5) = -\frac{1}{5!}, U(6) = 0,$$
$$U(7) = -\frac{1}{7!}, U(8) = 0 \ldots.$$

Therefore, from (5), the solution of the integral equation (12) is given by

$$u(y) = 1 - y - \frac{1}{3!}y^3 - \frac{1}{5!}y^5 - \frac{1}{7!}y^7 - \cdots$$
$$= 1 - \sinh(y),$$

which is the exact solution of the integral equation(12).

Example 3. Consider the nonlinear Volterra integral equation

$$u(y) + \int_0^y (u^2(t) + u(t))dt = \frac{3}{2} - \frac{1}{2}\exp(-2y). \;\; (14)$$

According to theorem 1 and to the operations of differential transformation given in table 1, we have the following recurrence relation:

$$U(k) + \sum_{l=0}^{k-1} U(l)\frac{U(k-l-1)}{k} + \frac{U(k-1)}{k} = \frac{3}{2}\delta(k) - \frac{(-2)^k}{2k!},$$
$$k \geq 1, U(0) = 1. \qquad (15)$$

Consequently, we find

$$U(1) = -1, U(2) = \frac{1}{2!},$$
$$U(3) = -\frac{1}{3!}, U(4) = \frac{1}{4!},$$
$$U(5) = -\frac{1}{5!}, U(6) = \frac{1}{6!}, \ldots$$

Therefore, from (5), the solution of the integral equation (14) is given by

$$u(y) = 1 - y + \frac{1}{2!}y^2 - \frac{1}{3!}y^3 + \frac{1}{4!}y^4 - \frac{1}{5!}y^5 + \cdots = \exp(-y),$$

which is the exact solution of the integral equation(14).

Example 4. Consider the nonlinear Volterra integral equation

$$u(y) = \cos(y) + \frac{1}{2}\sin(2y) + \; 3y - 2\int_0^y \big(1 + u^2(t)\big)dt. \qquad (16)$$

According to theorem 1 and to the operations of differential transformation given in the table 1, we have the following recurrence relation :

$$U(k) = \frac{1}{k!}\cos\left(\frac{\pi k}{2}\right) + \frac{2^{k-1}}{k!}\sin\left(\frac{\pi k}{2}\right) + \delta(k-1) - 2\sum_{l=0}^{k-1} U(l)\frac{U(k-l-1)}{k},$$
$$k \geq 1, \qquad U(0) = 1. \qquad (17)$$

Consequently, we find

$$U(1) = 0, U(2) = -\frac{1}{2!},$$
$$U(3) = 0, U(4) = \frac{1}{4!},$$
$$U(5) = 0, U(6) = -\frac{1}{6!},$$
$$U(7) = 0, U(8) = \frac{1}{8!}, \ldots$$

Therefore, from (5), the solution of the integral equation (16) is given by

$$u(y) = 1 - \frac{1}{2!}y^2 + \frac{1}{4!}y^4 - \frac{1}{6!}y^6 + \frac{1}{8!}y^8 + \cdots \;\; = \cos(y),$$

which is the exact solution of the integral equation(16).

IV. CONCLUSIONS

In this study, we apply the differential transformation method to the linear and nonlinear Volterra integral equations with separable kernels and find the exact solutions of the equations. This method reduces the computational diificulties of the other traditional methods and all the calculations can be made simple manipulations. Using the DTM, several examples were tested and the results have shown remarkable performance. therefore, this method can be applied to many nonlinear integral and differential equations without any linearization, discretization or perturbation.

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