Single Server Retrial Bulk Queue With Two Phases Of Essential Server And Vacation Policy

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Abstract: Consider a single server retrial queueing system in which bulk arrival follows Poisson process and the server provides two phases of heterogeneous essential service in succession with different types of vacation policies. By applying supplementary variable techniquewe derived Probability Generating Function(PGF) of number of customers in the retrial group. The performance measures *viz.*, the mean number of customers in orbit and mean waiting time in retrial queue obtained. Special cases also discussed.

Keywords: Retrial queue, Bulk arrival. Heterogeneous service, Server vacation, Supplementary variable technique, Probability Generating Function.

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1. INTRODUCTION

Queuing systems arise in the modelling of many practical applications related to communication, Production, computer net works and so on. The special characteristic of this system is that a customer who gains a busy server does not leave the system. He joins the retrial group from where he makes repeated attempts to obtain service. A new class of queueing system with repeated calls is characterized. At present however, most studies are devoted to batch arrival queues with vacation. Because of its interdisciplinary character considerable efforts have been devoted to study these models.

Queuing systems with batch arrivals are common in a number of real situations. In computer communication systems, message which are to be transmitted could consist of a random number of packets comparable works on optimal control policies for batch arrival case is seldom found in the literature. That motivates us to develop a realistic model for queueing system with batch arrivals.

Chaudhary and Templeton have provided a comprehensive review on bulk queues and their applications. Langaris and Moutzoukis have considered a retrial queue with structured batch arrivals preemptive resume priorities for a single vacation model.

Reni Sagayaraj and Moganraj (2015a,b) have considered Single server retrial queue starting failure, subject to break down with multiple vacations model.

In this paper,the single server retrial group queueing system has been considered with batch arrivals and general service. The steady state behavior of a $M^X/G/1$ retrial queueing system with two phases of service and different vacation policy have been analysed. Analytical treatment of this model is obtained using supplementary variable technique. The Probability Generating Function(PGF) of number of customers in the retrial group.

II. MATHEMATICAL MODEL

Consider single server retrial queueing system with batch arrival, The primary calls arrive in bulk according to a Poisson process with rate λ and the server provides two phases of heterogeneous service in succession. If the primary calls, on arrival, find the server busy then the arriving calls make an orbit and try to attempt the service. The server provides two essential services to all the arriving calls. As soon as possible the second service of the call is complete, the server may go for*i* th (*i*= 1,2,3..M) type of vacation with probability β. The time between two successive repeated attempts of each call in orbit is assumed to be exponentially distributed with rate v . Let

- $S_1(x)$ and $S_2(x)$ are cumulative distribution function of first essential and second essential services respectively.
- $S_1(x)$ and $S_2(x)$ are probability density functionsof first essential and second essential service respectively.
- $S_1(\theta)$ and $S_2(\theta)$ are Laplace Stieltjes Transforms of first essential and second essential services respectively.
- $S_1^0(x)$ and $S_2^0(x)$ are remaining service time of first essential and second essential services respectively.
- $V_i(x)$ and $[V_i^0(x)](i=,2,3,...M)$ are cumulative distribution functions of vacation times.
- N(t) is the number of customers in the orbit at time t .
- $X(z)=\sum_{k=1}^{\infty} g_k z^k$ is the generating function of the batch size distribution.

The states of the server are denoted as

$$C(t)=\begin{cases}0 & \text{If the server is idle}\\ 1 & \text{If the server is doing first essential service}\\ 2 & \text{If the server is doing second essential service}\\ 3 & \text{If the server is on vacation}\end{cases}$$

Define the following probability functions

$$P_{0,n}(t)dt=\Pr\{N(t)=n,C(t)=0\}\, n\geq 0$$

$$P_{i,n}(x,t)dt=\Pr\{N(t)=n,C(t)=i,x\leq S_i^0(t)\leq x+dt\};\ n\geq 0;\ i=1,2,3$$

III. STEADY STATE SYSTEM SIZE DISTRIBUTION

By applying supplementary variable technique, following equations are obtained for this queueing model,

$$P_{0,j}(t+\Delta t)=P_{0,j}(t)(1-\lambda\Delta t-j\nu\Delta t)+P_{3,j}(0,t)+\beta_0 P_{2,j}(0,t)\Delta t \tag{1}$$

$$P_{1,j}(x-\Delta t,t+\Delta t)=P_{1,j}(x,t)(1-\lambda\Delta t)+\lambda\Delta t\sum_{k=1}^{j+1}g_k P_{0,j-k+1}S_1(x)+(j+1)\nu P_{0,j+1}(0,t)S_1(x)\Delta t$$
$$+\lambda\Delta t\sum_{k=1}^{j}g_k P_{1,j-k}(x,t) \tag{2}$$

$$P_{2,j}(x-\Delta t,t+\Delta t)=P_{2,j}(x,t)(1-\lambda\Delta t)+\lambda\Delta t\sum_{k=1}^{j}g_k P_{2,j-k}(x,t)+P_{1,j}(0,t)S_2(x)\Delta t \tag{3}$$

$$P_{3,j}(x-\Delta t,t+\Delta t)=P_{3,j}(x,t)(1-\lambda\Delta t)+\lambda\Delta t\sum_{k=1}^{j}g_k P_{3,j-k}(x,t)+P_{2,j}(0,t)\sum_{k=1}^{M}B_k V_k(x)\Delta t \tag{4}$$

Steady state equations of the above equations are

$$(\lambda+j\nu)P_{0,j}=P_{3,j}(0)+\beta_0 P_{2,j} \tag{5}$$

$$-\frac{d}{dx}P_{1,j}(x)=-\lambda P_{1,j}(x)+\lambda\sum_{k=1}^{j+1}g_k P_{1,j-k+1}S_1(x)+(j+1)\nu P_{0,j+1}\,S_1(x)+\lambda\sum_{k=1}^{j}g_k P_{1,j-1}(x) \tag{6}$$

$$-\frac{d}{dx}P_{2,j}(x)=-\lambda P_{2,j}(x)+\lambda\sum_{k=1}^{j}g_k P_{2,j-k}(x)+P_{1,j}(0)\,S_1(x) \tag{7}$$

$$-\frac{d}{dx}P_{3,j}(x)=-\lambda P_{3,j}(x)+\lambda\sum_{k=1}^{j}g_k P_{3,j-k}(x)+P_{2,j}(0)\sum_{k=1}^{M}B_k V_k(x) \tag{8}$$

Define Laplace Steljes transform as

$$P_{ij}^{*}(x)=P_{ij}(\theta), i=0,1,2,3 \tag{9}$$

$$\theta P_{1,j}(\theta)-P_{1,j}(0)=\lambda P_{1,j}(\theta)-\lambda\sum_{k=1}^{j+1}g_k P_{0,j-k+1}S_1(\theta)-(J+1)\nu P_{0,j+1}S_1(\theta)-\lambda\sum_{k=1}^{j}g_k P_{1,j-k}(\theta) \tag{10}$$

$$\theta P_{2,j}(\theta)-P_{2,j}(0)=\lambda P_{2,j}(\theta)-\lambda\sum_{k=1}^{j}g_k P_{2,j-k}S_1(\theta)-P_{1,j}(0)S_2(\theta) \tag{11}$$

$$\theta P_{3,j}(\theta)-P_{3,j}(0)=\lambda P_{3,j}(\theta)-\lambda\sum_{k=1}^{j}g_k P_{3,j-k}(\theta)-P_{2,j}(0)\sum_{k=1}^{M}B_k V_k(\theta) \tag{12}$$

Now define the probability generating functions(PGF)

$$P_0(z)=\sum_{j=0}^{\infty}P_{0,j}z^j \tag{13}$$

$$P_i(z,\theta)=\sum_{j=0}^{\infty}P_{0,j}(\theta)z^j \tag{14}$$

and

$$P_i(z,0)=\sum_{j=0}^{\infty}P_{0,j}(0)z^j \tag{15}$$

Using the above started probability given function in the equations (5),(10),(11)and (12) which become

$$\lambda P_0(z) + \nu z P_0^1(z) = P_3(z,0) + \beta_0 P_2(z,0) \tag{16}$$

$$(\theta - \lambda + \lambda x(z)) P_1(z,\theta) = P_1(z,0) - \lambda \frac{x(z)}{z} P_0(z) S_1(\theta) - V P_0^1(z) S_1(\theta) \tag{17}$$

$$(\theta - \lambda + \lambda x(z)) P_2(z,\theta) = P_2(z,0) - P_1(z,0) S_2(\theta) \tag{18}$$

Substitute $\theta = \lambda + \lambda z$ in the equations (17) (18) and (19), and get,

$$P_1(z,0) = \frac{X(z)}{z} \lambda P_0(z) S_1(\lambda - \lambda X(z)) + \nu P_0^{'}(z) S_1(\lambda - \lambda X(z)) \tag{20}$$

$$P_2(z,0) = P_1(z,0) S_2(\lambda - \lambda X(z)) \tag{21}$$

$$P_3(z,0) = P_2(z,0) \sum_{k=1}^{M} B_k V_k (\lambda - \lambda X(z)) \tag{22}$$

The equation (16) is er-written after applying the equations(21)and(22),

$$\lambda P_0(z) + \nu z P_0^{'}(z) = \sum_{k=1}^{M} (\beta_k V_k (\lambda - \lambda X(z)) + \beta_0) S_2(\lambda - \lambda X(z)) \left(\lambda \frac{X(z)}{z} P_0(z) + \nu z P_0^{'}(z) \right) S_1(\lambda - \lambda X(z)) \tag{23}$$

On solving the linear differential equation (23) and get,

$$P_0(z) = P_0(1) \exp\left(\int_0^1 \frac{1 - [S_1(\lambda - \lambda X(u)) S_2(\lambda - \lambda X(u))]}{[S_1(\lambda - \lambda X(u)) S_2(\lambda - \lambda X(u))]} \right) \left(\frac{\sum_{k=1}^{M} (\beta_k V_k (\lambda - \lambda X(u)) + \beta_0) \frac{X(u)}{u}}{\sum_{k=1}^{M} (\beta_k V_k (\lambda - \lambda X(u)) + \beta_0) - u} \right) \tag{24}$$

Similarly, the equations (20)&(21) and (22) are used in the equations (17),(18) and (19) which respective become

$$(\theta - \lambda + \lambda X(z)) P_1(z,\theta) = S_1(\theta - \lambda + \lambda X(z)) - S_1(\theta) \left[\frac{X(z)}{z} \lambda P_0(z) + \nu P_0^{'}(z) \right] \tag{25}$$

$$(\theta - \lambda + \lambda X(z)) P_2(z,\theta) = P_1(z,\theta) \left(S_2(\lambda + \lambda X(z)) \right) - S_2(\theta) \tag{26}$$

and

$$(\theta - \lambda + \lambda X(z)) P_3(z,\theta) = P_2(z,0) \left(\sum_{k=1}^{M} \left(\beta_k (V_k (\lambda - \lambda X(z)) - V_k(\theta)) \right) \right) \tag{27}$$

Apply simple mathematics in the equations (25), (26) and (27) and respectively get

$$P_i(z,0) = \frac{\left(S_1(\lambda - \lambda X(z) - 1) \right) \left[\lambda P_0(z) \frac{X(z)}{z} + \nu P_0^{'}(z) \right]}{(-\lambda + \lambda X(z))} \tag{28}$$

$$P_2(z,0) = \frac{\left(S_1(\lambda - \lambda X(z) - 1) \right) S_2(\lambda - \lambda X(z) - 1) \left[\lambda P_0(z) \frac{X(z)}{z} + \nu P_0^{'}(z) \right]}{(-\lambda + \lambda X(z))} \tag{29}$$

$$P_3(z,0) = \frac{\left(S_1(\lambda - \lambda X(z) - 1) \right) S_2(\lambda - \lambda X(z) - 1) \left(\sum_{k=1}^{M} \left(\beta_k (V_k (\lambda - \lambda X(z)) - 1) \right) \right) \left[\lambda P_0(z) \frac{X(z)}{z} + \nu P_0^{'}(z) \right]}{(-\lambda + \lambda X(z))} \tag{30}$$

The Probability Generating Function P(z) of number of customers in the orbit at an arbitrary epoch can be expressed as

$$P(z) = P_0(z) + P_1(z,0) + P_2(z,0) + P_3(z,0) \tag{31}$$

Substituting the expressions of $P_0(z), P_1(z,0), P_2(z,0)$ and $P_3(z,0)$ in the equation (31) which yield

$$P(z)=\frac{\{1-R(z)+(R(z)-X(z))\}P_0(z)}{[R(z)-X(z)]\alpha} \qquad (32)$$

Where

$$R(z)=S_1(\lambda-\lambda X(z))S_2(\lambda-\lambda X(z))\left(\sum_{k=1}^{M}(\beta_k(V_k(\lambda-\lambda X(z))-\beta_0))\right) \qquad (33)$$

On considering $\lim_{z\to 1} p(z)=1$. The steady state condition is satisfied.

ie, $\rho=\lambda\left(E(S_1)+E(S_2)+\sum_{k=1}^{M}\beta_k E(V_k)\right)<1$

IV. PERFORMANCE MEASURE

Some useful results of our model are as follows

a) The mean number of customers in orbit

$$L=E[N(t)]=\underset{z\to 1}{lt}\frac{d}{dz}P(z)$$

$$P(z)=\frac{\{1-R(z)+(R(z)-X(z))\}P_0(z)}{[R(z)-X(z)]\alpha}$$

Where R(z)=$S_1(\lambda z-\lambda zX(z))$ and $P_0(z)=P_0(1)\exp\left\{\frac{-\lambda}{\nu}\int_z^1\left(\frac{1-R(u)}{R(u)-u}\right)du\right\}$

R(z)=$S_1(\lambda z-\lambda zX(z))(\beta_1 V_1(\lambda-\lambda X(z))+\beta_0)$

$$\nu_i(x)=\frac{(ku_i)^k x^{k-1}e^{-ku_i x}}{(k-1)!} \qquad (34)$$

Where $\gamma_2 E(S_1^2)+E(S_2^2)+\sum_{k=1}^{M}(\beta_k E[V_k^2]+2E[S_1]E[S_2]$

$$+2E[S_1]\sum_{k=1}^{M}(\beta_k E[V_k])+2E[S_2]\sum_{k=1}^{M}(\beta_k E[V_k])$$

b) Mean waiting time in retrial queue

We have the mean waiting time in the retrial queue(W) as follows.

$$W=\frac{L}{\lambda}$$

V. PARTICULAR CASES

Case: 1 If there is no second phase of service $S_2(\lambda z-\lambda zX(z)=1)$ and no vacation $V_k(\lambda z-\lambda zX(z))=1,\ k=1,2,3....M$ then the customers in orbit is given by

$$P(z)=\frac{\{1-R(z)+(R(z)-X(z))\}P_0(z)}{[R(z)-X(z)]\alpha} \qquad (35)$$

Where p(z)=$S_1(\lambda-\lambda X(z))$ and $P_0(z)=P_0(1)\exp\left\{\frac{-\lambda}{\nu}\int_z^1\left(\frac{1-[S_1(\lambda-\lambda(z))]}{S_1(\lambda-\lambda(z))-u}\right)du\right\}$

The above equation agrees the probability generating function of the customers in the orbit of M/G/1 retrial queue.

Case :2 If M=1 and no second phase of service $S_2(\lambda-\lambda X(z)=1)$, then the probability generating function of the number of customers in the orbit is given by

$$P(z)=\frac{\{1-z\}\,P_0(z)}{[R(z)-z]} \tag{36}$$

R(z)=$S_1(\lambda-\lambda(z))\left(\beta_1 V_1(\lambda-\lambda(z))+\beta_0\right)$ and $P_0(z)=P_0(1)\exp\left\{\frac{-\lambda}{\nu}\int_z^1\left(\frac{1-R(u)}{R(u)-u}\right)du\right\}$

Equating on (36) agree the probability generating function of the number of customers in the orbit of M/G/1 retrial queue in the steady state obtained.

Conclusion:

In this paper, a single server queue with two phases of essential services and different types of vacation policies is analyzed. We introduced the concepts of retrial queue and bulk arrivals while analyzing the queue. The system performance measure obtained explicitly and displayed to show the effects of the various parameters can be utilized to upgrade the concerned system during the development and design phase.

If one may apply *k* stages of heterogeneous essential services with sever breakdown in the existing model the researcher may get the another set of fruitful results.

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